Computation of multivariable signatures of colored links

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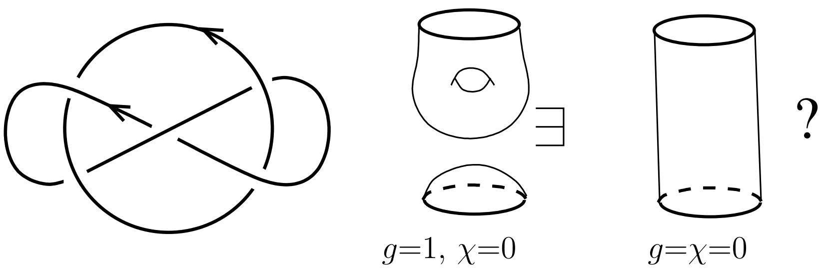
Definition. The *(smooth) slice genus* of a knot $K \subset S^3$ is $g_s(L) = \min_F \operatorname{genus}(F)$ where the minimum is taken over all smooth connected oriented surfaces in B^4 transverse to $S^3 = \partial B^4$ such that $\partial F = K$.

Definition. The *(smooth) slice Euler characteristic* of an oriented link $L \subset S^3$ is $\chi_s(L) = \max_F \chi(F)$ where the maximum is taken over all smooth oriented surfaces without closed components in B^4 transverse to $S^3 = \partial B^4$ and such that $\partial F = L$.

The problem of computation (or estimating) the slice genus of knots is one of the most classical problems of the knot theory. For knots we have $\chi_s(K) = 1 - 2g_s(K)$, thus it does not matter which one of these invariants to compute.

For links this is not the same. The computation of the slice genus is much harder and less useful (imho).

Example: $\chi_s(5^2_1) = 0$ but $g_s(5^2_1)$ seems to be unknown (it is given in `Linkinfo` but C.Livingston does not know why).



Estimation of χ_s has applications in algebraic geometry:

For example, Lee Rudolph gave a short and elegant knot-theoretic proof of the Abhyankar-Moh theorem:

if $X \subset \mathbb{C}^2$, $X \cong \mathbb{C}$, then $\exists\, \alpha \in \mathrm{Aut}(\mathbb{C}^2)$ s.t. $\alpha(X)$ is a line.

I used bounds for χ_s in the study of the topology of plane real algebraic curves (Hilbert's 16-th Pb.)

Definition. A *Seifert surface* of L is a smooth connected oriented embedded surface S in S^3 such that $\partial S = L$.

Definition. *Seifert form* is a (non-symmetric!) quadratic form α on $H_1(S)$ given by $\alpha(x, y) = \mathrm{lk}(x, y^+)$ where y^+ is the cycle y pushed off from S in the positive direction.

Definition. Let α be a quadratic form on V. *Witt index* $\mathrm{ind}(\alpha)$ is the max. dimension of a subspace U s.t. $\forall\, x, y \in U$, $\alpha(x, y) = 0$. *Witt coindex* of α is $\dim V - 2\,\mathrm{ind}(\alpha)$.

The simplest for computation and very efficient bounds of $\chi_s(L)$ come from the inequality

$$\chi_s(L) \leq 1 - (\text{Witt coindex of Seifert form of } L)$$

Upper bounds for the Witt coindex.

Murasugi–Tristram:

$$\forall\, z \in \mathbb{C} \setminus \{0\}, \quad \mathrm{coind}(\alpha) \leq \mathrm{signature}(z\alpha + \bar{z}\alpha^T)$$

Fox-Milnor:

$$\mathrm{coind}(\alpha) = 0 \;\Rightarrow\; \Delta(t) = \pm t^n F(t) F(t^{-1}),\; F \in \mathbb{Z}[t]$$

where $\Delta(t) = \det(\alpha - t\alpha^T)$ (this is the Alexander polynomial of L when α is a Seifert form of L)

In particular, the Fox-Milnor condition implies $\Delta(-1) = \pm n^2$ and $\Delta(\sqrt{-1}) = \pm(m^2 + n^2)$ for $m, n \in \mathbb{Z}$

There are also arithmetic bounds for $\mathrm{coind}(\alpha)$ obtained by reducing coefficients of α mod p,

Colored links

A μ-*colored link* is a link whose components are colored in μ colors, more formally, a link with a fixed presentation as a disjoint union of sublinks $L = L_1 \cup \cdots \cup L_\mu$.

Similarly, a μ-*colored surface* is a surface with a fixed presentation as a disjoint union of subsurfaces $F = F_1 \cup \cdots \cup F_\mu$.

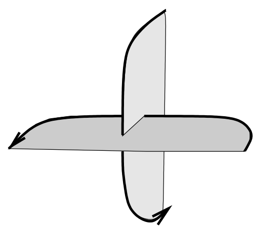
Definition. The *(smooth) slice Euler characteristic* of an oriented μ-colored link $L = L_1 \cup \cdots \cup L_\mu$ is $\chi_s(L) = \max_F \chi(F)$ where the maximum is taken over all smooth oriented μ-colored surfaces $F = F_1 \cup \cdots \cup F_\mu$ without closed components in B^4 transverse to $S^3 = \partial B^4$ and such that $\partial F_i = L_i$ for each $i = 1, \ldots, \mu$.

The problem of estimating χ_s of μ-colored links naturally arises in my study of real alg. curves (Hilbert's 16th Pb.)

The following estimate of $\chi_s(L)$ and the underlying constructions is a synthesis of several papers by David Cimasoni, Daryl Cooper, Vincent Florens, and Oleg Viro.

Definition. Let $L = L_1 \cup \cdots \cup L_\mu$ be a μ-colored link.
A *C-complex* of L is a union $C = C_1 \cup \cdots \cup C_\mu$ of embedded oriented pairwise transverse surfaces such that:

- $\partial C_i = L_i$, for any i;
- $C_i \cap C_j \cap C_k = \varnothing$ for any distinct i, j, k;
- $C_i \cap C_j$ is a union of clasps for any i, j (each clasp is an embedded segment connecting L_i to L_j).

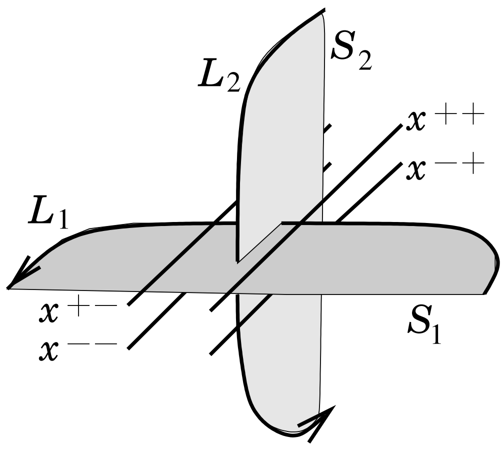


Exercise. Prove that any colored link admits a C-complex.

Definition. A *(colored) Seifert form* of an oriented μ-colored link L is the quadratic form α on $H_1(C)$ with values in $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ whose value on $x, y \in H_1(C)$ is

$$\alpha(x, y) = \sum_{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\mu) \in \{-1,1\}^\mu} \operatorname{lk}(x, y^\varepsilon)\, t_1^{\varepsilon_1} \ldots t_\mu^{\varepsilon_\mu}$$

where, for $x \in H_1(C)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\mu) = (\pm 1, \ldots, \pm 1)$, we define x^ε as a loop close to C and disjoint from C obtained from x by pushing it off from each C_i in the direction ε_i



Estimate for $\chi_s(L)$ via Witt coindex. For any μ-colored link L and for the Seifert form α on any C-complex of L, one has

$$\chi_s(L) \geq 1 - \operatorname{coind}(\alpha).$$

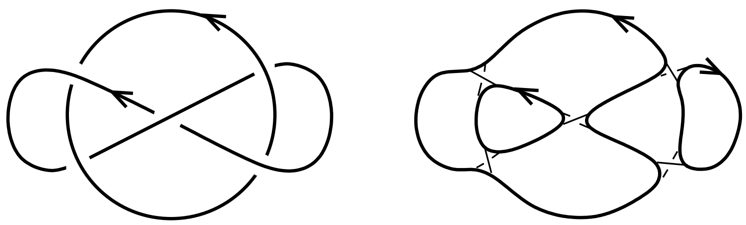
In particular, we have analogs of Murasugi-Tristram inequality and Fox-Milnor condition. Their ingredients are colored link invariants. These are the multivariable signatures (see the title of this talk) and the multivariable Alexander polynomial.

Remark 1. If one considers a usual link L as a 1-colored link, then the colored Seifert form is $t\alpha + t^{-1}\alpha^T$ where α is the usual Seifert form. Clearly, the Witt coindex of these forms is the same.

C O M P U T A T I O N S

First recall the classical Seifert's algorithm for a Seifert surface starting from a link diagram.

- Remove the crossings according to the orientations
- Fill the circles (called *Seifert circles*) by disjoint disks
- Join the disks by twisted ribbons (see the picture)



A computation of a Seifert matrix based on Seifert's algorithm is implemented in Bar Natan's `KnotTheory` package (can be easily hacked from the `Alexander` function).

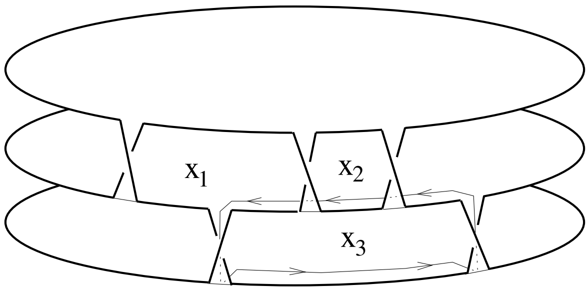
I will present a version of this algorithm for braids.
and its modification for **colored braids** and thus for colored links.

Why braids?

- In my way to do real alg. geom, links appear as braids
- Easy to encode the input
- For me, it was more easy to implement

We start with my Seifert matrix algorithm for usual (non colored) braids.

The Seifert surface via Seifert circles is particularly easy to see (and to encode!) for braids:

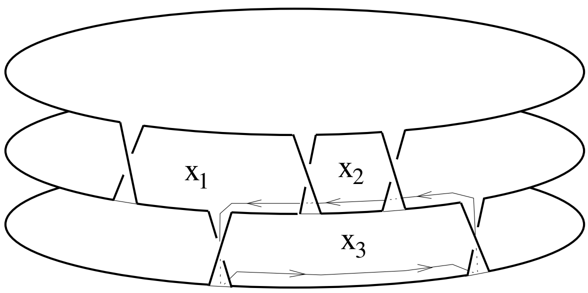


We see immediately a base of $H_1(S)$. Its elements are determined just by some pairs of crossings.

We also see in this picture that each crossing σ_i contributes to $\alpha(x, y)$ only when x and y belong to the (at most) 4 cycles adjacent to it: from left, right, above, and below.

Thus, up to technical details, the algorithm is as follows:

Scan the crossings from left to right and at each crossing modify the $16 = 4 \times 4$ *entries of the Seifert matrix which correspond to* $\alpha(x, y)$ *for the adjacent cycles* x *and* y.



The Mathematica code implementing this algorithm:

```
SeifertMatrix=Function[{m,brd},
Module[{n,e,V,X,q,c,i,j,h,a,b},
  a={{ 0,1,-1,0},{-1, 0,1,0},{0,0,0,0},{1,-1,0,0}};
  b={{-1,1, 0,0},{ 1,-1,0,0},{0,0,0,0},{0, 0,0,0}};
  n=Length[brd]; V=Table[0,{i,n},{j,n}];
  X=Table[{n},{h,m-1}];
  Do[ h=Abs[brd[[q]]]; e=Sign[brd[[q]]];
    c[1]=X[[h,1]]; X[[h]]={c[2]=q};
    c[3]=If[h<m-1,X[[h+1,1]],n];
    c[4]=If[h>1,X[[h-1,1]],n];
    Do[Do[ V[[ c[i],c[j] ]] += a[[i,j]]+e*b[[i,j]],
   {i,4}],{j,4}],
  {q,n}];
  Transpose[Delete[Transpose[Delete[V,X]],X]]/2 ]];
```

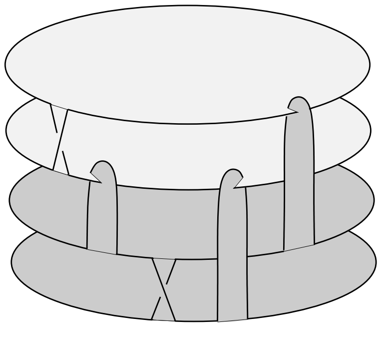
Now let's pass to the colored version.

A *colored braid* is a braid whose strings are colored so that the the colors of left and right endpoints match to each other. We also assume that strings of the same color are consecutive.

Proposition. *A colored link can be given by a colored braid which is a product of σ_i's for monochrome crossings and* pure braid group generators $A_{i,j}$ *(squares of the band generators).*

Then the C-complex C is:

A basis of $H_1(C)$ is not as evident as for usual braid braids



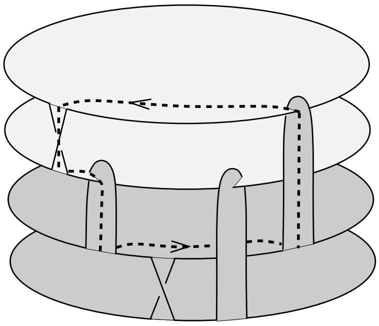
Definition. *Special A-word* is a word in σ_i's and A_{ij}'s such that for any its prefix (i.e. initial subword) of the form wA_{ij}, the word w contains $\sigma_k^{\pm 1}$ or $A_{k,k+1}^{\pm 1}$ for each $k = i, \dots, j-1$.

To make an A-word special, one can start it with

$$(A_{1,2}A_{1,2}^{-1})(A_{2,3}A_{2,3}^{-1}) \dots\dots (A_{n-1,n}A_{n-1,n}^{-1})$$

Example of a special A-word:

Here we see a base of $H_1(C)$. These cycles:

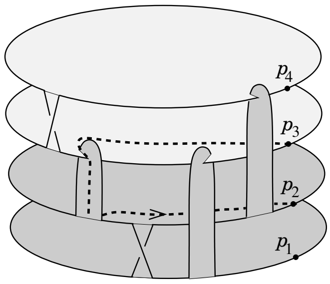


But these cycles are not so easy to manipulate.
Even to encode them. So, we use the following trick.

We successively compute the Seifert form on $H_1(C, P)$ where

- $C = C^{(m)}$ is the C-complex for the m-th prefix;
- $P = P^{(m)} = \{p_1, \dots, p_n\}$ is as in this picture:

We choose a base of $H_1(C, P)$ composed of the old (absolute) and new (relative) cycles, these ones:



At the end of computation we just delete the lines and columns corresponding to the relative cycles.

COMPLEXITY ISSUES

Recall that μ is the number of colors. It is clear that for any fixed μ, the algorithm is polynomial (with small degree) with respect to the length of the input A-word.

At the first glance, the algorithm (even its output) should be exponential with respect to μ. Indeed, recall that

$$\alpha(x, y) = \sum_{\varepsilon=(\varepsilon_1,\dots,\varepsilon_\mu)\in\{-1,1\}^\mu} \operatorname{lk}(x, y^\varepsilon)\, t_1^{\varepsilon_1} \dots t_\mu^{\varepsilon_\mu}$$

(2^μ summands).

The output is indeed of exponential length if we expand all the polynomials in $t_1^{\pm 1}, \dots, t_\mu^{\pm 1}$. However...

However, surprisingly the length of the output as well as the running time of the above algorithm becomes polynomial in μ if we do not expand.

Namely, a good news is that the contributions of each letter $\sigma_i^{\pm 1}$ of a special A-word is of the form

$$c \prod_{j \in J} (t_j - t_j^{-1}), \qquad J \subset \{1, \dots, \mu\}$$

and the contribution of A_{ij} is a sum of at most $j - i$ terms of this form.

But a bad news is...

A bad news is that I do not know how to convert any given braid word W to a special A-word whose length is polynomial simultaneously w.r.t. $length(W)$ and w.r.t. μ.

Hope that one of you will find such a polynomial algorithm.

Thank you for your attention