On amphicheiral links

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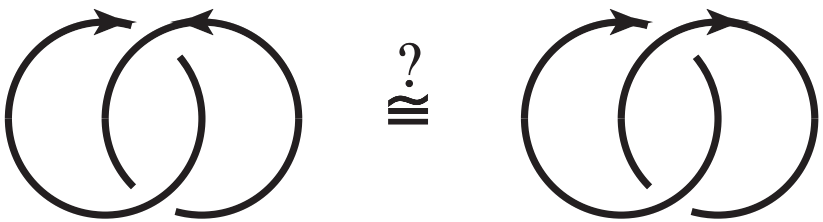
The link symmetric group, the linking graph, and a tangle sum presentation of an amphicheiral link
in preparations

Contents

§1. Main Theorem

amphicheiral link = link equivalent to its mirror image.

amphicheiral = achiral,   non-amphicheiral = chiral.



Problems

Problem 1. Determine a link is amphicheiral or not.

Problem 2. Study invariants of amphicheiral links.

Problem 3. Is the minimal crossing number of an amphicheiral link always even ?

Answer : No in general, but Yes for alternating links.

For every odd $c \geq 15$, there exists an amphicheiral knot with the minimal crossing number c [Stoimenow].

$^{\exists}2$-component amphicheiral link with the minimal crossing number $9, 11$ [K-Kawauchi], [K].

Problem 4. Is there general construction of amphicheiral links ?

(Is any amphicheiral link constructed by a special tangle sums ?)

We made a table of prime amphicheiral links with $\sharp$ of components ≥ 2 and the crossing number ≤ 11. cf. Knot Atlas

Conjecture 4.11. [K]
L : n-component algebraically split component-preservingly amphicheiral link. n : even $\Longrightarrow \Delta_L(t_1, \ldots, t_n) = 0$.

$L = K_1 \cup \cdots \cup K_n$: algebraically split $\Longleftrightarrow$
$^{\forall}\ell_{ij} = \mathrm{lk}(K_i, K_j) = 0 \quad (1 \leq i < j \leq n)$.

Theorem 4.12. [K-Kawauchi]
$L = K_1 \cup \cdots \cup K_n$: n-component amphicheiral link.

$$\ell_{ij} = \mathrm{lk}(K_i, K_j), \quad n + \sum_{1 \le i < j \le n} \ell_{ij} \text{ : even}$$

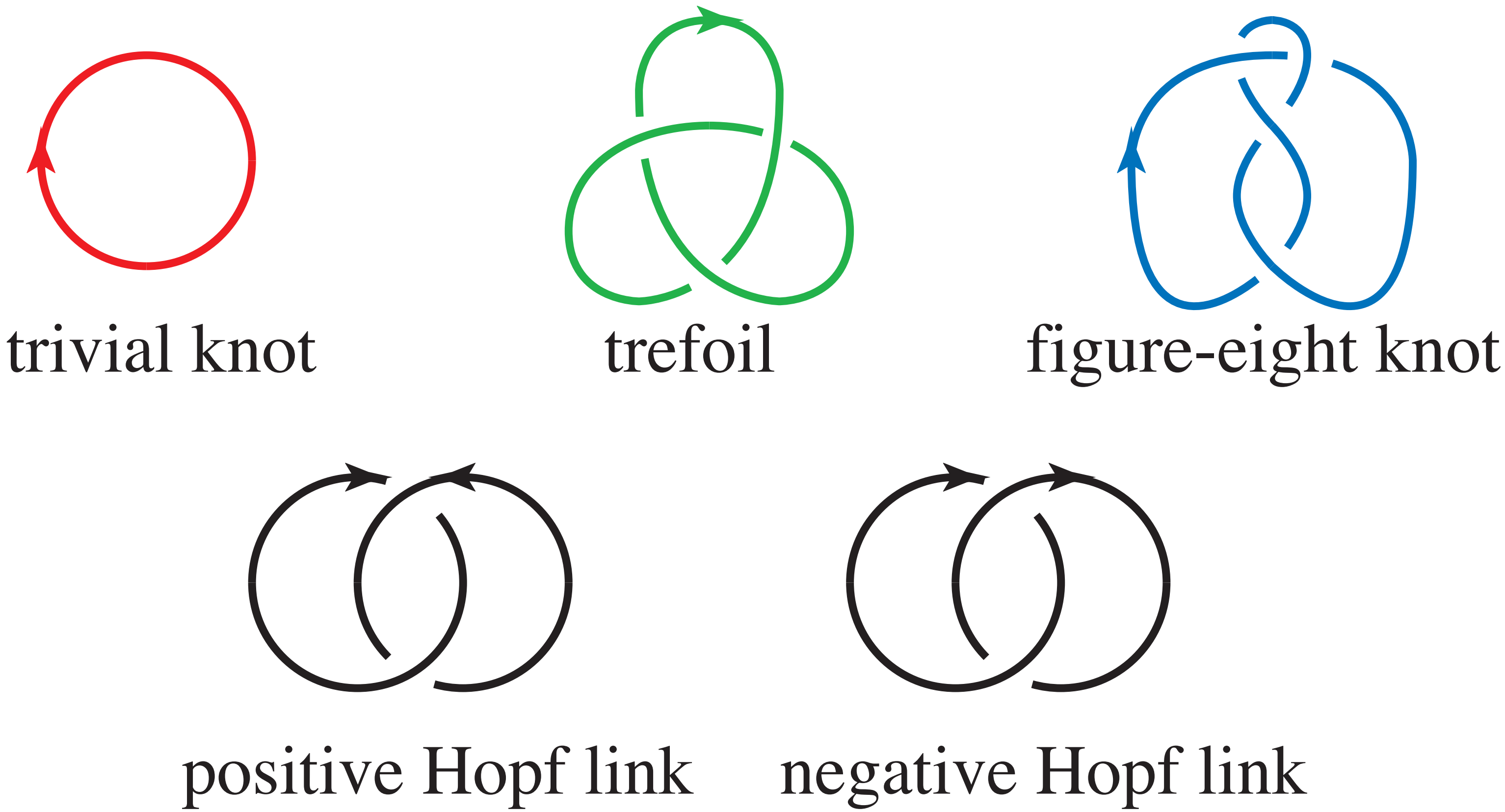
$\Longrightarrow \Delta_L(-1, \ldots, -1) = 0$.

Theorem 4.13. [K-Kawauchi]
L : n-component component-preservingly (ε)-amphicheiral link. n : even $\Longrightarrow \Delta_L(t_1, \ldots, t_n) = 0$.

[Traldi] shows the cases
(1) $n = 2$ & $\varepsilon = \pm$, and (2) $n \geq 3$ & $\varepsilon = -$.

§2. Amphicheiral/Invertible link

Link



$f : \coprod_{i=1}^{n} (S^1)_i \hookrightarrow S^3$: embedding

$L = (S^3, \mathrm{Im}(f))$: n-component link in S^3
$K_i = (S^3, f((S^1)_i))$: the i-th component of L

$$L = K_1 \cup \cdots \cup K_n$$

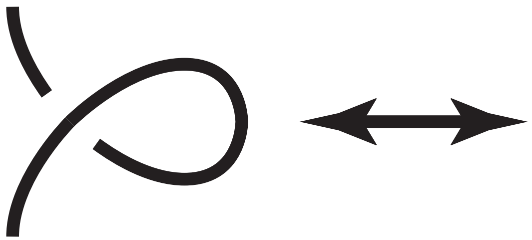
L : oriented link $\Longleftrightarrow$ $^{\forall}i$, K_i : oriented
L : unoriented link $\Longleftrightarrow$ $^{\forall}i$, K_i : unoriented

L : ordered link $\Longleftrightarrow$ The order of the indices $(1, 2, \ldots, n)$ is fixed.
L : unordered link $\Longleftrightarrow$ L is not ordered.

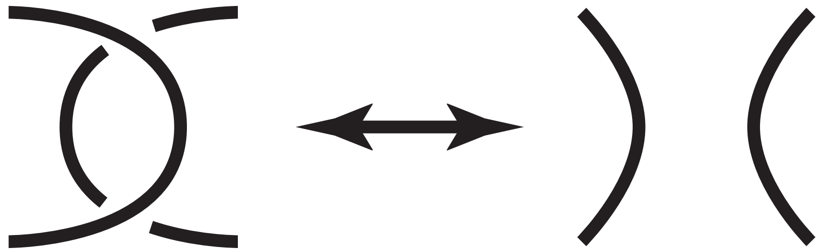
• $L,\ L'$: equivalent $\quad L \cong L'$
$\Longleftrightarrow$ $^{\exists}h : S^3 \to S^3$: orientation-preserving homeo. s.t. $L' \cong h(L)$
(as oriented/unoriented/ordered/unordered links).

Reidemeister moves

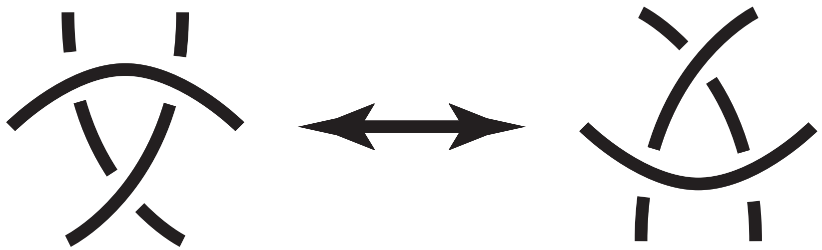
(R1)



(R2)



(R3)



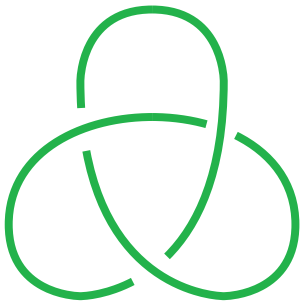
$p : S^3 = \mathbb{R}^3 \cup \{\infty\} \to S^2 = \mathbb{R}^2 \cup \{\infty\}$: natural projection
$p((x, y, z)) = (x, y)$, $p(\infty) = \infty$.

$D = p(L)$: link diagram of L

Theorem 2.1. (Fundamental Theorem)
${}^\forall D,\ D'$: diagrams of L
D and D' are related by a finite sequence of Reidemeister moves.

$\{$classical links$\} = \{$classical link diagrams$\}/\langle$(R1), (R2), (R3)$\rangle$

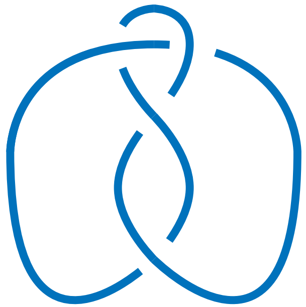
Amphicheiral link



3_1

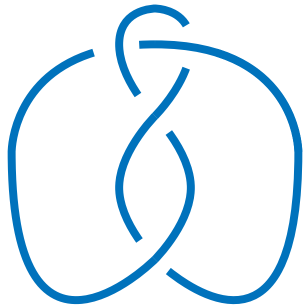
$\overset{?}{\cong}$

3_1^*

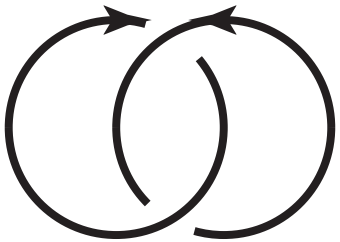


4_1

$\overset{?}{\cong}$

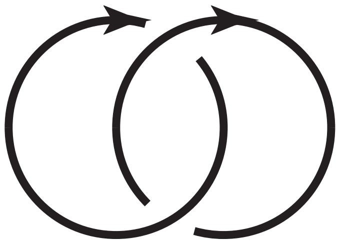


4_1^*



2_1^2

$\overset{?}{\cong}$



2_1^{2*}

L : link in S^3

D : diagram of L

$h : S^3 \to S^3$: orientation-reversing homeomorphism

$L^* = h(L)$: mirror image of L

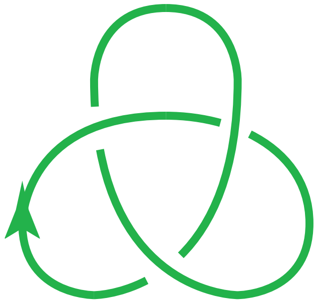
Since $\mathrm{MCG}(S^3) = \{\iota, \tau\} \cong \mathbb{Z}/2\mathbb{Z}$, we can take $h = \tau$.
$(\tau((x, y, z)) = (x, y, -z))$

$D^* = p \circ \tau(L)$: mirror image of D

$\circ$ D^* : diagram of L^*

L : amphicheiral link $\Longleftrightarrow$ $L \cong L^*$ as unoriented links.

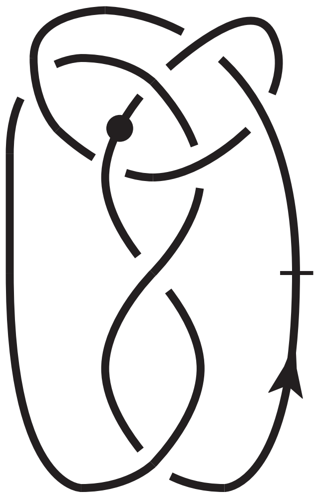
Invertible link



3_1

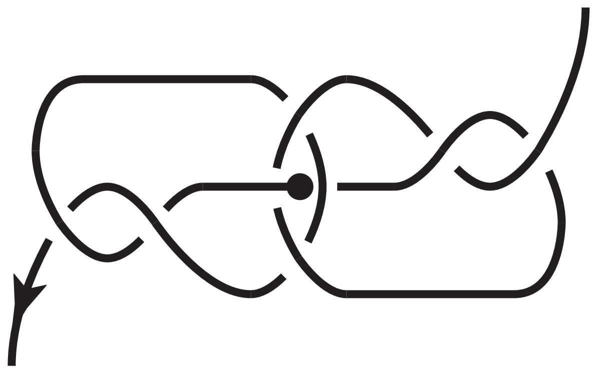
$\overset{?}{\cong}$

-3_1



8_{17}

$\cong$



K : oriented knot

$-K$: orientation-reversed knot of K

$L = K_1 \cup \cdots \cup K_n$: oriented link

$-L = (-K_1) \cup \cdots \cup (-K_n)$: orientation-reversed link of L

L : invertible link $\Longleftrightarrow$ $L \cong -L$ as oriented links.

- K : oriented knot $\Longrightarrow$ $K \sharp K^*$, $K \sharp (-K^*)$: amphicheiral knots.
- K : oriented knot $\Longrightarrow$ $K \sharp (-K)$: invertible knot.

§3. Link symmetric group

$L = K_1 \cup \cdots \cup K_n$: n-component link
Fix the orientation of S^3, and the orientation and the order of L.

$N = \{\varphi \in \mathrm{MCG}(S^3, L) \mid \varphi$: ori.-pres. homeo. of S^3 preserving ori.s & ord. of $L\} \triangleleft \mathrm{MCG}(S^3, L)$

$\Gamma(L) = \mathrm{MCG}(S^3, L)/N$: link symmetric group of L

[Whitten], [Hillman]

Then $\Gamma(L)$ is determined by

∘ ori.-pres./rev. homeo. of $S^3 \in \mathrm{MCG}(S^3) \cong \{+1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$.
∘ orientations of $K_1, \ldots, K_n \in \{+1, -1\}^n \cong (\mathbb{Z}/2\mathbb{Z})^n$.
∘ The order of the components $\in \mathfrak{S}_n$.

$[f] \in \Gamma(L)$, $f(K_i) = \varepsilon_{\sigma(i)} K_{\sigma(i)}$ $(\sigma \in \mathfrak{S}_n,\ \varepsilon_i \in \{+1, -1\})$
f : ori.-pres. homeo. of $S^3 \Longrightarrow L$: $(\varepsilon_1, \ldots, \varepsilon_n; \sigma)$-invertible
f : ori.-rev. homeo. of $S^3 \Longrightarrow L$: $(\varepsilon_1, \ldots, \varepsilon_n; \sigma)$-amphicheiral

$\Gamma_n = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^n \times \mathfrak{S}_n$: the universal set.
We denote $(\kappa, (\varepsilon_1, \ldots, \varepsilon_n), \sigma) \in \Gamma_n$ by $\mathbf{x} = \kappa(\varepsilon_1, \ldots, \varepsilon_n; \sigma)$.
We introduce a group structure into Γ_n.
Γ_n : the n-th universal link symmetric group
$\Gamma(L) \subset \Gamma_n$: the link symmetric group of L

(group operation)

$\mathbf{x} = \kappa(\varepsilon_1, \ldots, \varepsilon_n; \sigma)$, $\mathbf{y} = \zeta(\eta_1, \ldots, \eta_n; \tau) \in \Gamma_n$,

$\mathbf{y} \cdot \mathbf{x} = \kappa\zeta(\varepsilon_{\tau^{-1}(1)}\eta_1, \ldots, \varepsilon_{\tau^{-1}(n)}\eta_n; \tau \circ \sigma)$

(associativity)

$\mathbf{z} = \xi(\lambda_1, \ldots, \lambda_n; \rho) \in \Gamma_n$,

$\mathbf{z} \cdot \mathbf{y} \cdot \mathbf{x} = \kappa\zeta\xi\cdot$

$(\varepsilon_{\tau^{-1}\circ\rho^{-1}(1)}\eta_{\rho^{-1}(1)}\lambda_1, \ldots, \varepsilon_{\tau^{-1}\circ\rho^{-1}(n)}\eta_{\rho^{-1}(n)}\lambda_1; \rho \circ \tau \circ \sigma)$

$\mathbf{1} = +(+, \ldots, +; \iota)$

$\mathbf{x}^{-1} = \kappa(\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(n)}; \sigma^{-1})$

◦ We also denote by

$+(\varepsilon_1, \ldots, \varepsilon_n; \sigma) = (\varepsilon_1, \ldots, \varepsilon_n; \sigma)$, $\pm(\varepsilon_1, \ldots, \varepsilon_n; \iota) = \pm(\varepsilon_1, \ldots, \varepsilon_n)$,

$\pm(\varepsilon, \ldots, \varepsilon; \sigma) = \pm(\varepsilon)$.

$\Gamma_n^0 = \{(\varepsilon_1, \ldots, \varepsilon_n; \sigma)\} \subset \Gamma_n$

: the n-th universal $(+)$-link symmetric group $\boxed{\Gamma_n \cong \mathbb{Z}/2\mathbb{Z} \times \Gamma_n^0}$

$\pi : \Gamma_n^0 \to \mathfrak{S}_n$: natural surjection $\quad \mathrm{Ker}(\pi) \cong (\mathbb{Z}/2\mathbb{Z})^n$

$1 \to (\mathbb{Z}/2\mathbb{Z})^n \xrightarrow{i} \Gamma_n^0 \xrightarrow{\pi} \mathfrak{S}_n \to 1$: split exact

$\mathbf{x} = (\varepsilon_1, \ldots, \varepsilon_n)$, $\mathbf{y} = (+, \ldots, +; \sigma) \in \Gamma_n$,

$\mathbf{y}^{-1} \cdot \mathbf{x} \cdot \mathbf{y} = (\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(n)})$

$\varphi : \mathfrak{S}_n \to \mathrm{Aut}\left((\mathbb{Z}/2\mathbb{Z})^n\right)$

$\varphi(\sigma) = ((\varepsilon_1, \ldots, \varepsilon_n) \mapsto (\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(n)}))$

$\boxed{\Gamma_n^0 \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes_\varphi \mathfrak{S}_n}$ and $\boxed{\Gamma_n \cong \mathbb{Z}/2\mathbb{Z} \times ((\mathbb{Z}/2\mathbb{Z})^n \rtimes_\varphi \mathfrak{S}_n).}$

∘ $|\Gamma_n| = 2^{n+1} \cdot n!$, $\quad |\Gamma_n^0| = 2^n \cdot n!$.

- $\Gamma^0(L) = \Gamma(L) \cap \Gamma_n^0$: the $(+)$-link symmetric group of L

Determination problem

Determine $\Gamma(L) \subset \Gamma_n$.

Realization problem

Which subgroups of Γ_n can be realized as $\Gamma(L)$?

- $\Gamma_1^0 \cong \mathbb{Z}/2\mathbb{Z}, \quad \Gamma_1 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

- $a = (+, -; (1\ 2))$, $b = (+, +; (1\ 2))$

$\Gamma_2^0 = \langle a, b \mid a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle \cong D_4$.

Link symmetries [Whitten], [Hillman]

$\mathbf{x} = \kappa(\varepsilon_1, \ldots, \varepsilon_n; \sigma) \in \Gamma(L) \Longrightarrow L$: $\mathbf{x}$-symmetric

- $\sigma = \iota \Longrightarrow L$: component-preservingly $\mathbf{x}$-symmetric

$\mathbf{x} \in \Gamma^0(L) \Longrightarrow L$: $(\varepsilon_1, \ldots, \varepsilon_n; \sigma)$-invertible

- ${}^\forall \varepsilon_i = - \Longrightarrow L$: $(-)$-invertible
- $\Gamma^0(L) \setminus \{\mathbf{1}\} \neq \emptyset \Longleftrightarrow L$: invertible

$\mathbf{x} \in \Gamma(L) \setminus \Gamma^0(L) \Longrightarrow L$: $(\varepsilon_1, \ldots, \varepsilon_n; \sigma)$-amphicheiral

- ${}^\forall \varepsilon_i = \varepsilon \Longrightarrow L$: (ε)-amphicheiral
- $\Gamma(L) \setminus \Gamma^0(L) \neq \emptyset \Longleftrightarrow L$: amphicheiral

§4. Conditions from invariants

Milnor's $\bar{\mu}$-invariant

$L = K_1 \cup \cdots \cup K_n$: oriented ordered n-component link

$\bar{\mu}_L(\cdots)$: Milnor's $\bar{\mu}$-invariant

∘ $\bar{\mu}_L(ij) = \mathrm{lk}(K_i, K_j)$ $(i \neq j)$: linking number

$\Psi_n : \Gamma_n \to \mathbb{Z}/2\mathbb{Z}$

$\mathbf{x} = \kappa(\varepsilon_1, \ldots, \varepsilon_n; \sigma) \in \Gamma(L)$,

$$\Psi_n(\mathbf{x}) = \kappa^{n-1} \left(\prod_{i=1}^{n} \varepsilon_i \right) \mathrm{sign}(\sigma)$$

$\Omega_n = \mathrm{Ker}(\Psi_n)$: the n-th Ω-subgroup

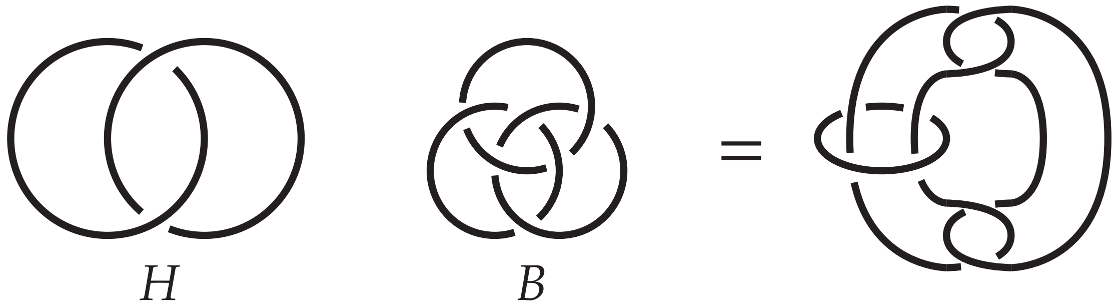
∘ $[\Gamma_n : \Omega_n] = 2$.

- $\Omega_1 = \{+(+), -(+)\} \cong \mathbb{Z}/2\mathbb{Z}$.

- $c = +(-,+;(1\ 2)),\ d = -(+,+;(1\ 2)) \in \Gamma_2$
$\Omega_2 = \langle c, d \mid c^4 = d^2 = 1,\ d^{-1}cd = c^{-1} \rangle \cong D_4$.

- $s = (-,+;(1\ 2)),\ t = (+,+;(1\ 2)) \in \Gamma_3^0$
$\Omega_3^0 = \Omega_3 \cap \Gamma_3^0 = \langle s, t \mid s^3 = t^4 = (st)^2 = 1 \rangle \cong \mathfrak{S}_4$.
$\Omega_3 = \mathbb{Z}/2\mathbb{Z} \times \Omega_3^0 \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4$.

Theorem 4.1. $n \geq 2,\ \bar{\mu}_L(12\cdots n) \neq 0 \Longrightarrow \Gamma(L) \subset \Omega_n \subset \Gamma_n$.



- $\Gamma(H) = \Omega_2,\ \Gamma(B) = \Omega_3$.

Theorem 4.2. $L = K_1 \cup \cdots \cup K_n$, $\ell_i = \mathrm{lk}(K_i, K_{i+1})$
$(i = 1, \ldots, n; K_{n+1} = K_1)$.
(1) n : odd & ${}^\forall \ell_i \neq 0$
$\Longrightarrow L$: not component-preservingly amphicheiral.

(2) $n = 3$ & ${}^\forall \ell_i \neq 0 \Longrightarrow L$: non-amphicheiral.

Theorem 4.3. $L = K_1 \cup K_2$, $\ell = \mathrm{lk}(K_1, K_2)$.
(1) [Hartley]
L : component-preservingly amphicheiral $\Longrightarrow \ell = 0$ or odd.

(2) L : $(\varepsilon, -\varepsilon; (1\ 2))$-amphicheiral $\Longrightarrow \ell \not\equiv 2 \pmod 4$.

Alexander polynomial

$L = K_1 \cup \cdots \cup K_n$: oriented ordered n-component link

$\Delta_L(t_1, \ldots, t_n) \in \mathbb{Z}\left[t_1^{\pm 1}, \ldots, t_n^{\pm 1}\right]$: Alexander polynomial of L

It is determined up to multiplication of $\pm t_1^{m_1} \cdots t_n^{m_n}$.

$A,\ B \in \mathbb{Z}\left[t_1^{\pm 1}, \ldots, t_n^{\pm 1}\right]$, $A \doteq B \iff A = \pm t_1^{m_1} \cdots t_n^{m_n} B$.

Theorem 4.4. (Duality) $\Delta_L(t_1, \ldots, t_n) \doteq \Delta_L\left(t_1^{-1}, \ldots, t_n^{-1}\right)$.

Lemma 4.5. L : $\kappa(\varepsilon_1, \ldots, \varepsilon_n; \sigma)$-symmetric link

$\Longrightarrow \Delta_L(t_1, \ldots, t_n) \doteq \Delta_L\left(t_{\sigma(1)}^{\varepsilon_{\sigma(1)}}, \ldots, t_{\sigma(n)}^{\varepsilon_{\sigma(n)}}\right)$.

∘ Useless for component-preservingly (ε)-symmetric links.

Theorem 4.6. [Hartley-Kawauchi]

(1) K : $(-)$-amphicheiral knot

$\Longrightarrow {}^{\exists} f(t) \in \mathbb{Z}[t]$ s.t. $f\left(t^{-1}\right) \doteq f(-t)$ & $|f(1)| = 1$ &

$\Delta_K(t^2) \doteq f(t) f\left(t^{-1}\right) \doteq f(t) f(-t)$.

(2) K : $(+)$-amphicheiral knot

$\Longrightarrow {}^{\exists} r_j(t) \in \mathbb{Z}[t]$: type X, ${}^{\exists}\alpha_j > 0$: odd $(j = 1, \ldots, m)$ s.t.

$$\Delta_K(t) \doteq \prod_{j=1}^{m} r_j\left(t^{\alpha_j}\right).$$

K : hyperbolic $\Longrightarrow m = \alpha_1 = 1$.

$r(t) \in \mathbb{Z}[t]$: type $X \iff$
${}^{\exists}k \geq 0$ & $\lambda \geq 3$: odd & $g_i(t) \in \mathbb{Z}[t]$ $(i = 0, \ldots, k)$ s.t.

$g_i(t) \doteq g_i(t)^{2^i} p_\lambda(t)^{2^{i-1}} \pmod 2$ $(i > 0)$ where

$$p_\lambda(t) = \frac{t^\lambda - 1}{t - 1} \ \& \ r(t) \doteq \begin{cases} g_0(t)^2 & (k = 0) \\ g_0(t)^2 g_1(t) \cdots g_k(t) & (k \geq 1). \end{cases}$$

Corollary 4.7. K : $(-)$-amphicheiral knot
$|\Delta_K(-1)| = p_1^{r_1} \cdots p_m^{r_m}$: prime factorization
${}^{\exists}p_i \equiv 3 \pmod 4 \implies r_i$: even.

∘ $|\Delta_{3_1}(-1)| = 3 \implies 3_1$: not $(-)$-amphicheiral.
3_1 : invertible $\implies 3_1$: not amphicheiral.

∘ $\Delta_{8_{17}}(t) \doteq t^6 - 4t^5 + 8t^4 - 11t^3 + 8t^2 - 4t + 1$
& 8_{17} : $(-)$-amphicheiral $\implies 8_{17}$: not invertible.

Theorem 4.8. [K]
L : n-component algebraically split component-preservingly (ε)-amphicheiral link.
n : even $\Longrightarrow \Delta_L(t^{\eta_1}, \dots, t^{\eta_n}) = 0$ $(\eta_i \in \{1, -1\})$.

Theorem 4.9. [K]
L : 2-component algebraically split component-preservingly amphicheiral link. $\Longrightarrow (t_1 - 1)^2(t_2 - 1)^2 | \Delta_L(t_1, t_2)$.

Theorem 4.10. [K]
L : 2-component algebraically split (ε)-amphicheiral link.
$\Longrightarrow (t_1 - 1)^2(t_2 - 1)^2(t_1 t_2 - 1)(t_1 - t_2) | \Delta_L(t_1, t_2)$.

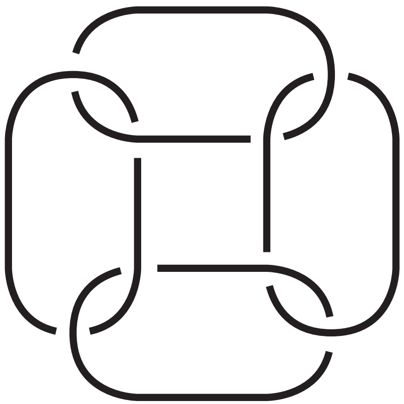
Conjecture 4.11. [K]
L : n-component algebraically split component-preservingly amphicheiral link. n : even $\Longrightarrow \Delta_L(t_1, \ldots, t_n) = 0$.

Theorem 4.12. [K-Kawauchi]
$L = K_1 \cup \cdots \cup K_n$: n-component amphicheiral link.

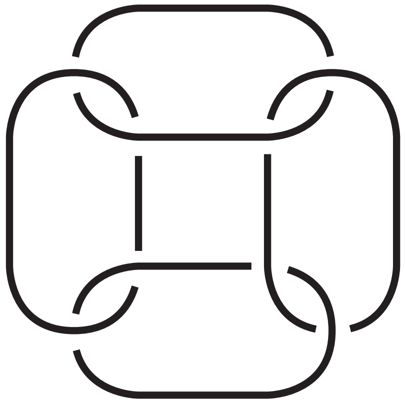
$\ell_{ij} = \mathrm{lk}(K_i, K_j)$, $\quad n + \sum_{1 \leq i < j \leq n} \ell_{ij}$: even

$\Longrightarrow \Delta_L(-1, \ldots, -1) = 0$.

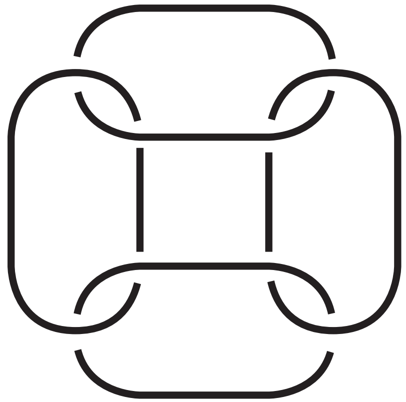
Theorem 4.13. [K-Kawauchi]
L : n-component component-preservingly (ε)-amphicheiral link. n : even $\Longrightarrow \Delta_L(t_1, \ldots, t_n) = 0$.



8^4_1
not amphicheiral



8^4_2
not amphicheiral



8^4_3
amphicheiral

$\Delta_{8^4_1}(-1,-1,-1,-1) = \pm 16 \Longrightarrow$: not amphicheiral

$\Delta_{8^4_2}(-1,-1,-1,-1) = \pm 16 \Longrightarrow$: not amphicheiral

$\Delta_{8^4_3}(-1,-1,-1,-1) = 0.$

Jones polynomial

L : oriented n-component link

$V_L(t) \in \mathbb{Z}\left[t^{\pm\frac{1}{2}}\right]$: Jones polynomial of L

Theorem 4.14.

L : amphicheiral $\Longrightarrow$ The coefficients of $V_L(t)$ are symmetric.

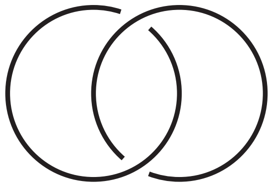
$\circ$ $V_{3_1}(t) = -t^4 + t^3 + t \Longrightarrow 3_1$: non-amphicheiral.

Theorem 4.15. [K-Kawauchi], [K]
The following is a list of prime amphicheiral links with ♯ of components ≥ 2 and the crossing number ≤ 11. The notation is from Knot Atlas.

2^2_1	6^2_2	6^3_2	8^2_8	8^3_4	8^3_6	8^4_3
9^2_{61}	10^2_{a56}	10^2_{a81}	10^2_{a83}	10^2_{a86}	10^2_{a116}	10^2_{a120}
10^2_{a121}	10^3_{a136}	10^3_{a140}	10^3_{a151}	10^3_{a156}	10^3_{a158}	10^4_{a169}
10^2_{n36}	10^2_{n46}	10^2_{n59}	10^4_{n105}	10^4_{n107}	11^2_{n247}	

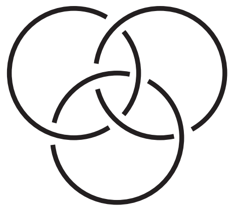
blue = not component-preservingly amphicheiral

Prime amphicheiral links up to 11 crossings (1) [K-Kawauchi], [K]



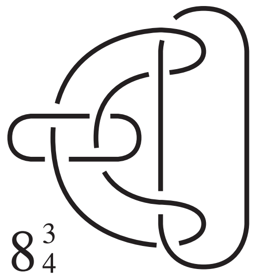
2^2_1

6^2_2

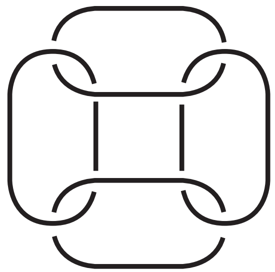


6^3_2

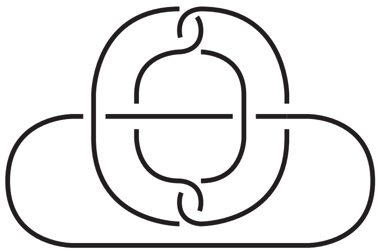
8^2_8



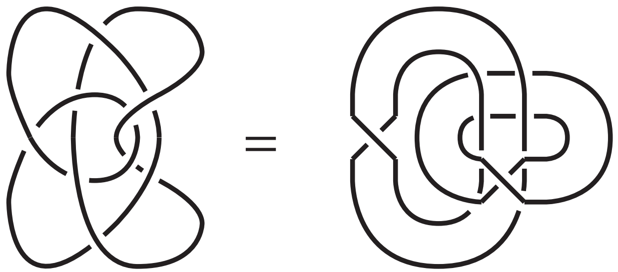
8^3_4



8^4_3

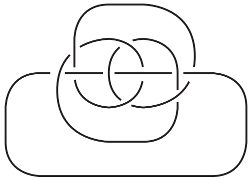


8^3_6

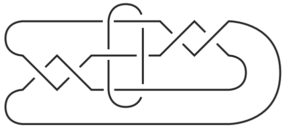


9^2_{61}

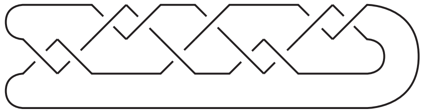
Prime amphicheiral links up to 11 crossings (2) [K-Kawauchi], [K]



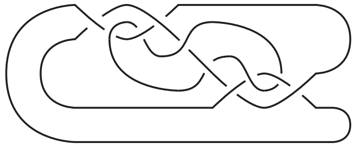
10^2_{a56}



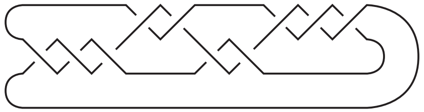
10^2_{a81}



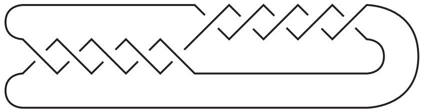
10^2_{a83}



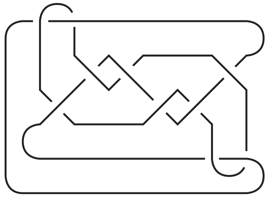
10^2_{a86}



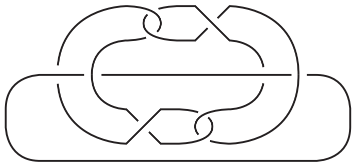
10^2_{a116}



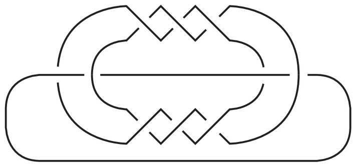
10^2_{a120}



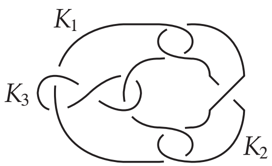
10^2_{a121}



10^3_{a136}

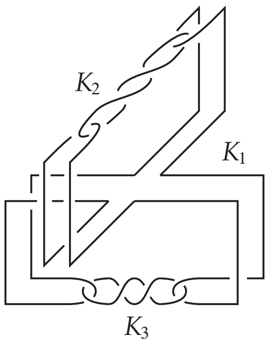


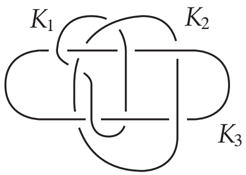
10^3_{a140}



10^3_{a151}

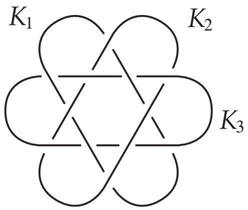
=



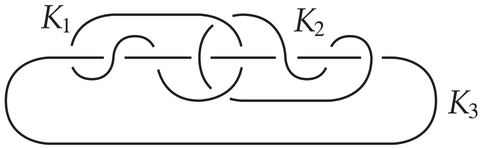


10^3_{a156}

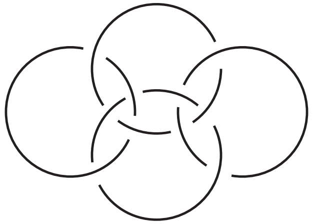
=



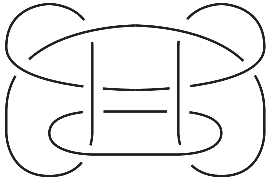
Prime amphicheiral links up to 11 crossings (3) [K-Kawauchi], [K]



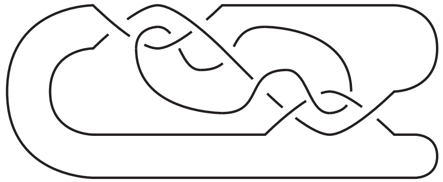
10^3_{a158}



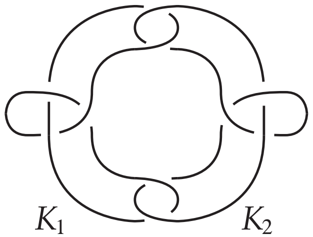
10^4_{a169}



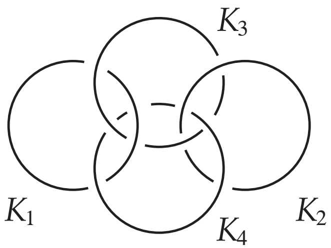
10^2_{n36}



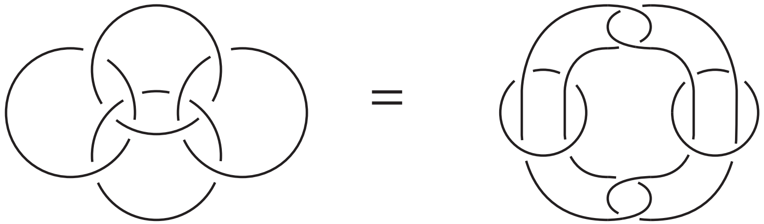
10^2_{n46}



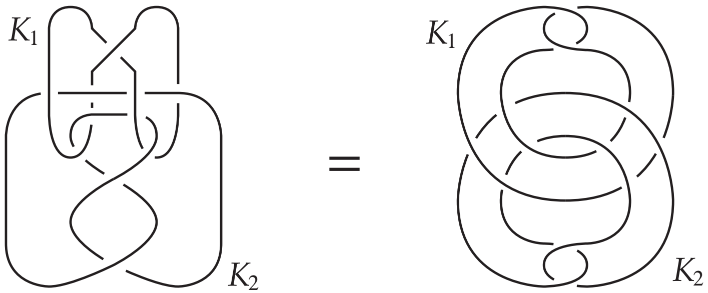
10^2_{n59}



10^4_{n105}



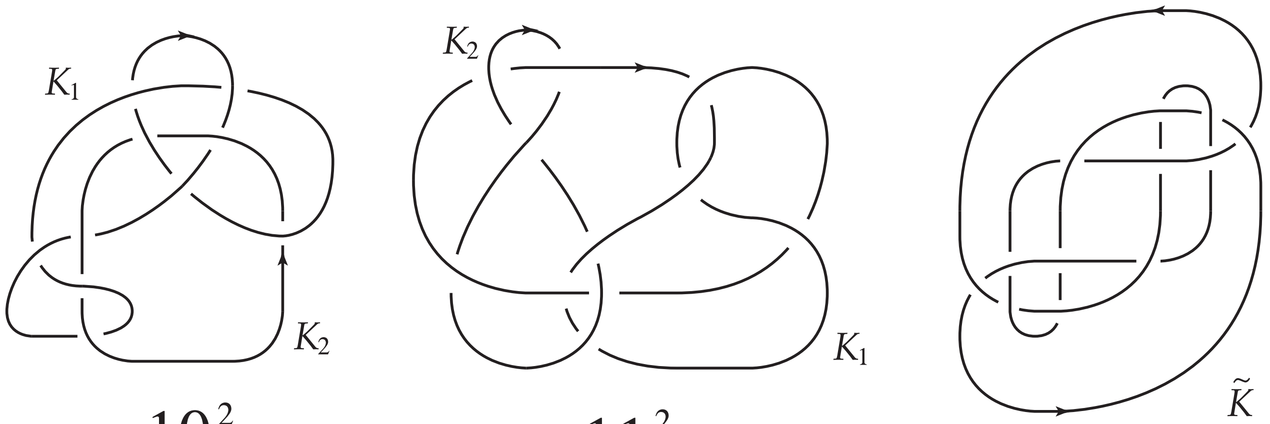
10^4_{n107}



11^2_{n247}

The HOMFLY polynomial does not detect that 10^2_{a51} and 11^2_{n127} are not amphicheiral.

The 2-variable Alexander polynomial detects that 10^2_{a51} is not amphicheiral.

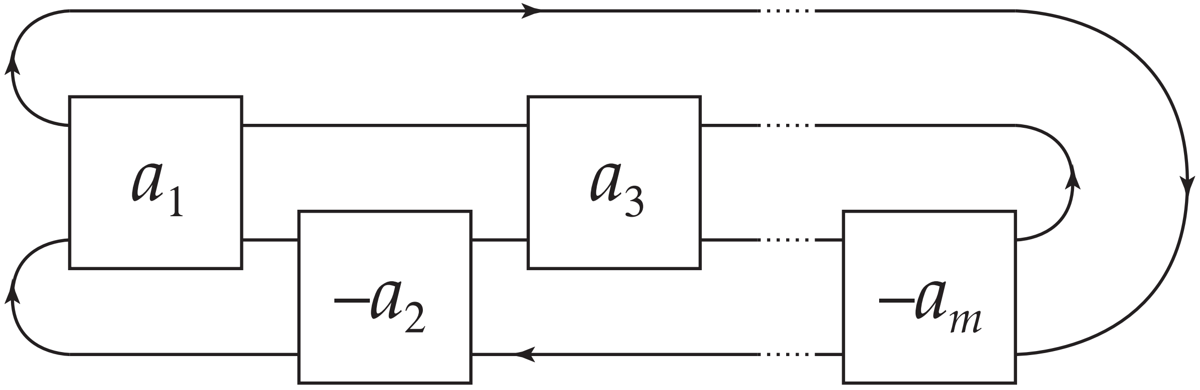


10^2_{a51} 11^2_{n127}

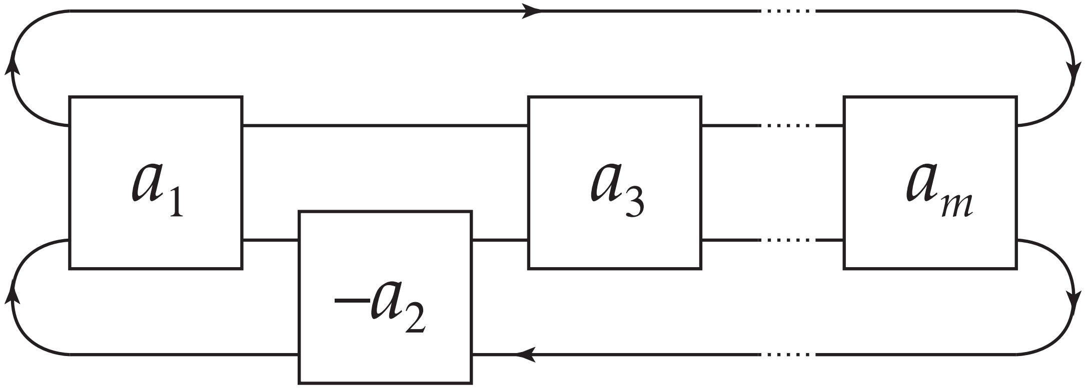
§5. Tangle sum construction of amphicheiral links

2-bridge link

$C(a_1, \ldots, a_m)$: Conway's notation



m: even



m: odd

$S(\alpha,\beta)$: Schubert's notation

$$[a_1,\ldots,a_m] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{\ddots}{a_{m-1} + \cfrac{1}{a_m}}}} = \frac{\alpha}{\beta}$$

$\Longleftrightarrow C(a_1,\ldots,a_m) \cong S(\alpha,\beta)$.

Theorem 5.1.

(1) $K = S(\alpha, \beta)$ with α : odd (2-bridge knot) is amphicheiral

$\iff \beta^2 \equiv -1 \pmod{\alpha}$.

$\iff {}^{\exists}a_1, \ldots, a_m$ with m : even

s.t. $K = C(a_1, \ldots, a_m)$ with $a_i = a_{m+1-i}$.

(2) $L = S(\alpha, \beta)$ with α : even (2-comp. 2-bridge link) is amphicheiral

$\iff \beta^2 + \alpha \equiv -1 \pmod{2\alpha}$.

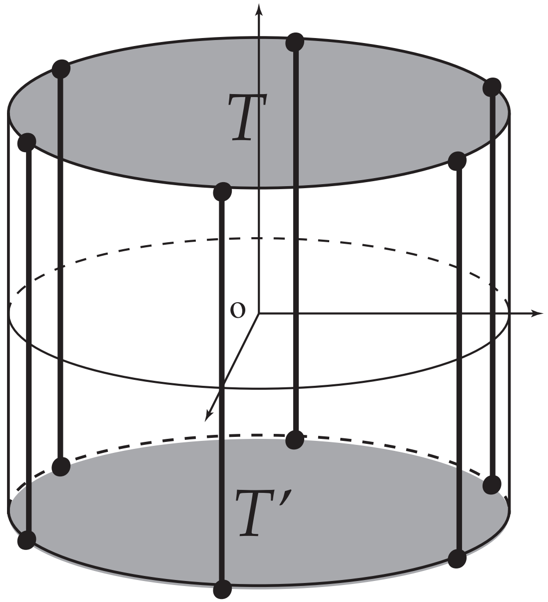
$\iff {}^{\exists}a_1, \ldots, a_m$ with m : even

s.t. $L = C(a_1, \ldots, a_m)$ with $a_i = a_{m+1-i}$.

Tangle sum

T, T' : tangles with n arcs & some loops

$L = C[T, T']$: tangle sum of L.



$C[T, T']$

some actions to T

T

T^*

$T \cdot \tau$

$T \cdot \rho$

$$D_m = \langle \tau, \rho \mid \tau^2 = \rho^m = \iota,\ \tau^{-1}\rho\tau = \rho^{-1} \rangle$$
$$= \{\tau^\eta \rho^k \mid \eta \in \mathbb{Z}/2\mathbb{Z} = \{0,1\},\ k \in \mathbb{Z}/m\mathbb{Z}\}$$

D_m acts on the complex unit disk as

τ : complex conjugate

ρ : rotation with $2\pi/m$ degree

T^* : mirror image of T

T, T' : tangles with n arcs (& some loops)

$x, y \in D_{2n}$

$C[T,T';x,y] := C[T\cdot x, T'\cdot y] \cong C[T, T'\cdot yx^{-1}] = C[T,T';\iota, yx^{-1}]$.

$T(\eta,k) := C[T,T^*;\iota,\tau^\eta\rho^k]$.

$T(k) = T(0,k), \quad T_-(k) = T(1,k)$.

Lemma 5.2. [K-Kobatake]

T : tangle with n arcs $\Longrightarrow$

$T(0),\ T(n),\ T_{-}(k)\ (k \in \mathbb{Z}/2n\mathbb{Z})$: strongly amphicheiral.

If L has an expression $T(\eta, k)$, then

$T(\eta, k)$: a tangle sum construction of L.

Tangle sum construction problem

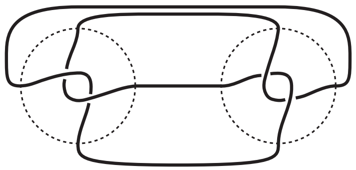
For an amphicheiral link, is it made by a tangle sum construction ?

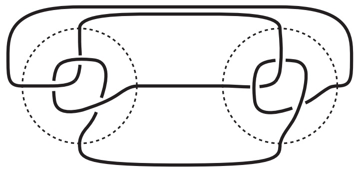
Theorem 5.3. [K-Kobatake]

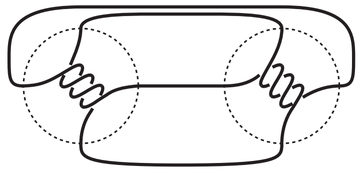
$^{\forall}$prime amphicheiral knot with the minimal crossing number ≤ 12,

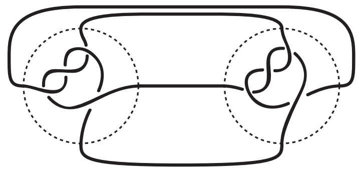
$^{\forall}$prime amphicheiral link with the minimal crossing number ≤ 11

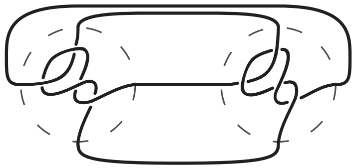
are made by tangle sum constructions.

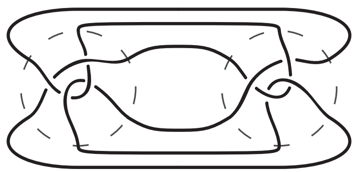

4_1

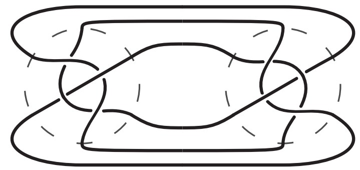

6_3

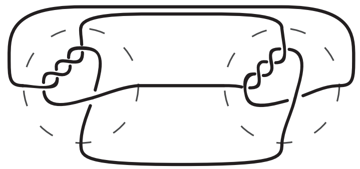

8_3

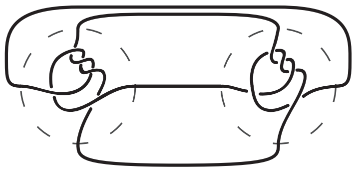

8_9

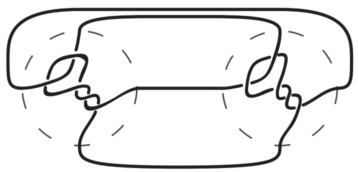

8_{12}

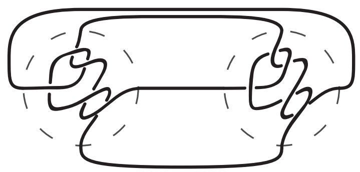

8_{17}

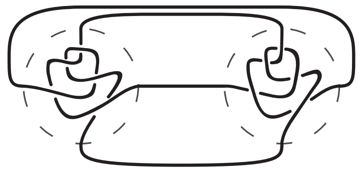

8_{18}

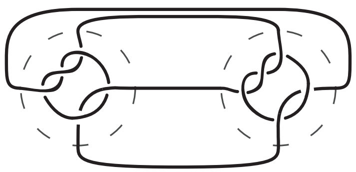

10_{17}

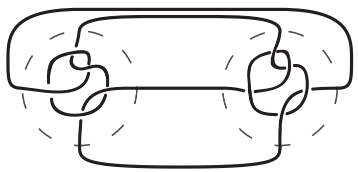

10_{33}

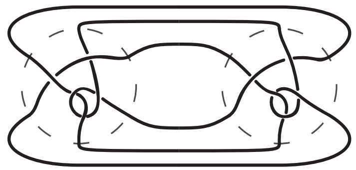

10_{37}

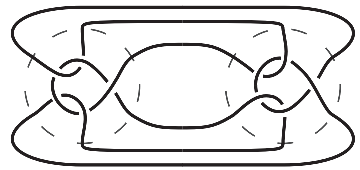

10_{43}


10_{45}


10_{79}


10_{81}


10_{88}


10_{99}

§6. Future Study

(1) Define and study amphicheirality for links in general 3-manifolds (including the case of virtual links).

(2) Use more refined invariants such as Heegaard Floer homology, Khovanov homology, etc.

(3) Study relationship with the cosmetic surgery problem.

(4) Study the tangle sum construction problem. Is it affirmative for prime amphicheiral links ? Is there counterexample in general ?
cf. even crossing number problem (Problem 3)

(5) Find fundamental moves among different tangle sum constructions of an amphicheiral link.

Thank you for your attention !