EXAMPLES OF KNOTS WITHOUT MINIMAL STRING BENNEQUIN SURFACES

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Abstract. Bennequin showed that any link of braid index 3 has a minimal genus Seifert surface placed naturally on a 3-braid, and Rudolph that any link has such a surface for some, not necessarily minimal, braid representation. With explicit examples, we close the gap between both results and show that Bennequin's theorem does not extend to knots of braid index 4.

1. Introduction

Let L be a link positioned as the closure $\hat{\beta}$ of a braid $\beta \in B_s$ with s strings (or strands) with respect to the axis A. A Seifert surface S for L is called a *braided Seifert surface* with respect to A if S consists of s parallel disks and some bands connecting them such that
(1) A penetrates the disks in the same direction each in a point, and
(2) each band is described by

$$[i,j]^{\pm 1} := \sigma_i \dots \sigma_{j-2}\sigma_{j-1}^{\pm 1}\sigma_{j-2}^{-1}\dots\sigma_i^{-1}\,, \tag{1}$$

for $1 \le i < j \le s$, where σ_i's are Artin's standard generators.

In particular, Birman-Menasco called braided Seifert surfaces S, which have minimal genus among all Seifert surfaces of the link L, *Bennequin surfaces*.

For any representation of a braid $\beta \in B_s$ as a word in $[i,j]^{\pm 1}$ one obtains a braided Seifert surface of $L = \hat{\beta}$ by inserting disks for each braid string and connecting them by half-twisted bands (the "embedded bands" in Rudolph's terminology) along each $[i,j]^{\pm 1}$. Rudolph showed that any (in particular, minimal genus) Seifert surface is isotopic to a braided Seifert surface [11], for a representation with a possibly very large number of strings. Bennequin [1] proved that for any link L of braid index 3, there exists a Bennequin surface for L with respect to a 3-string braid. This was later extended by Birman-Menasco (c.f. [2, p. 31], [16]), showing that for any such link L, *any* minimal genus surface is a Bennequin surface with respect to a 3-string braid. Generalizations of these results would be of interest, since they have several applications. They were of central importance for the classification of 3-braid links [2] (using a development of Bennequin's own methods) and of alternating links of braid index 3 [14] (using methods similar to the ones that will be applied here. Note that the latter classification is not contained in former, since from the method of [2] it is unclear which of the links are alternating.) Then one is naturally led to the question whether a Bennequin surface can always be obtained from a minimal string band representation. That is, for knots one asks: if g is the genus of a knot K with braid index $b > 3$, does K have a b-string braid representation with $2g+b-1$ bands? In this paper we give examples that this is not the case.

Theorem 1 There exist knots of braid index 4 whose 4-string braids never carry a Bennequin surface.

The examples we present are of genus $g = 3$. They were found using a construction of exotic Seifert surfaces and an extensive analysis of the skein polynomial P [4]. This procedure is explained in §2. The most substantial part of our argument is to show that 4-braid representations with 9 bands do not realize any of these knots. In the original computation, in which the examples were found, we checked representatives of all 4-braids that can be written

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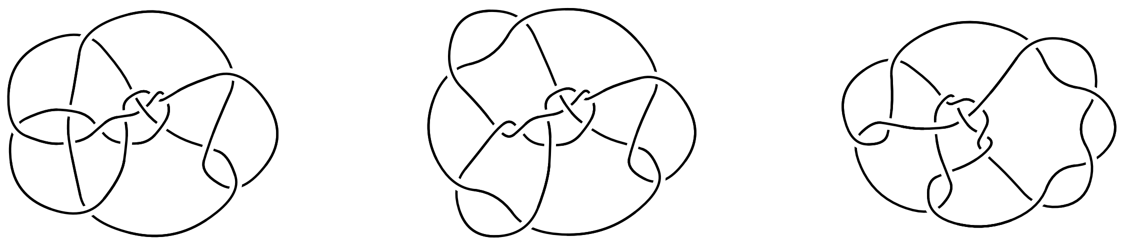
with (at most) 9 bands. Using efficiently band relations, the number of such representatives can be reduced to a manageable magnitude. In §3 then we consider the three specific examples given in §2 and justify their correctness. For this purpose we use some properties of the skein polynomial on these three knots. Then we prove an inequality for the (Alexander variable) degree of the polynomial on 4-braid band representations, which reduces the number of braids to check to a minimum. We additionally describe in detail our method in §2, so that there is a clear and easy way to verify our examples. In §4 we conclude with an example related to the Morton-Short paper [9], namely one for which even the 2-cabled inequality of Morton-Williams-Franks fails to determine the braid index. This knot came out in an attempt to refine our previous examples.

A while after this work was completed, and briefly before its publication, we were pointed to a similar example of a 2-component (boundary) link found by Ko and Lee in 1997. Their approach is quite different, in that they use to a much larger extent band word and conjugacy algorithms, and then apply an argument using braid foliations.

2. Examples



16_{606453} 16_{644823} $16_{1102350}$

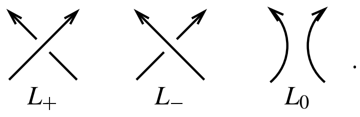
```
16 606453      0     10
 -2  4     -3    -7    -7    -2
 -2  4      3    10     7     1
 -2  4      3     7    10     4
 -2  4     -4   -15   -16    -4
 -2  4      1     7     7     1
  0  2           -1    -1
```

```
16 644823      0     10
 -4  2      2     5     5     1
 -4  2     -7   -21   -18    -5
 -4  2      9    33    30     8
 -4  2     -5   -24   -23    -5
 -4  2      1     8     8     1
 -2  0           -1    -1
```

```
16 1102350      0     10
 -2  2           -1    -1    -1
 -4  2     -4   -10    -7    -2
 -4  2      8    27    24     7
 -4  2     -5   -23   -22    -5
 -4  2      1     8     8     1
 -2  0           -1    -1
```

Figure 1: Three of the examples of Theorem 1 and their skein polynomials. All three knots have the Alexander polynomial $\Delta = -1\ 4\ -8\ [11]\ -8\ 4\ -1$.

We will use henceforth the following variable convention for P: $v^{-1}P(L_+) + vP(L_-) = -zP(L_0)$. Here as usual $L_{\pm,0}$ denote links with diagrams equal except near one crossing, which is resp. *positive*, *negative* and *smoothed out* (nullified):



This convention can be obtained from the one of [6] by setting $v = l^{-1}$, $z = m$. It *differs* from the one of [7], although we use the same variables. The inequalities of [7, 3] apply in the same way, though. We will denote by $\min\deg_v P$ resp. $\max\deg_v P$ the *minimal* resp. *maximal degree* of v in P, and by $\operatorname{span}_v P$ their difference.

Consider the knots 16_{606453}, 16_{644823} and $16_{1102350}$ in the tables of [5] shown on Figure 1. These knots (considered in the sequel up to taking the mirror image) have the following 4-braid band representations:

$$\begin{array}{rl} 16_{1102350}: & [1,2]^{-1}[2,4]^{-2}[1,2][1,3]^{-1}[1,2]^{-1}[2,4][1,2]^2[1,3]^2 \\ 16_{644823}: & [1,2]^{-1}[2,4]^{-2}[1,2][1,3]^{-1}[1,2]^{-1}[2,4][1,2][2,3]^{-1}[1,3]^2 \\ 16_{606453}: & [1,2]^{-1}[2,4]^{-2}[1,2][1,3]^{-1}[1,2]^{-1}[2,4][1,2][1,3]^2[2,3] \end{array} \qquad (2)$$

All three examples have an identical Alexander polynomial of degree 3, and hence they have genus $g \geq 3$. We see that they have genus 3, by actually spanning a surface by "tube compression". See Figure 2 (b) for the example $16_{1102350}$. (The figures work similarly for all three examples.) In the genus 4 surfaces obtained from the band representations in (2), the two holes created by the two pairs of letters $[1,2]^{\pm 1}$, through which the bands of the letters $[2,4]^{\pm 1}$ pass, can be compensated by adding a tube containing the bands of the letters $[2,4]^{\pm 1}$ and the disk of the fourth string, thus giving a surface of genus 3.

Since $\text{span}_v P = 6$ for these examples, we see that they have braid index $b = 4$ by the inequality of Morton-Williams-Franks [7, 3]: $b \geq {}^1\!/_2\, \text{span}_v P + 1$. However, it will be shown that no prime knot with a 4-string band representation of nine bands has the skein polynomial of any of them. They are seen to have unknotting number one (by changing $[1,2]^{-1}[2,4]^{-2}\cdots$ to $[1,2]^{-1}[2,4][2,4]^{-1}\cdots$), and hence are prime by [13].

Remark 1 Through the isotopy in Figure 2 (b) – (d), we see that the three examples have a Bennequin surface on a 5-string braid.

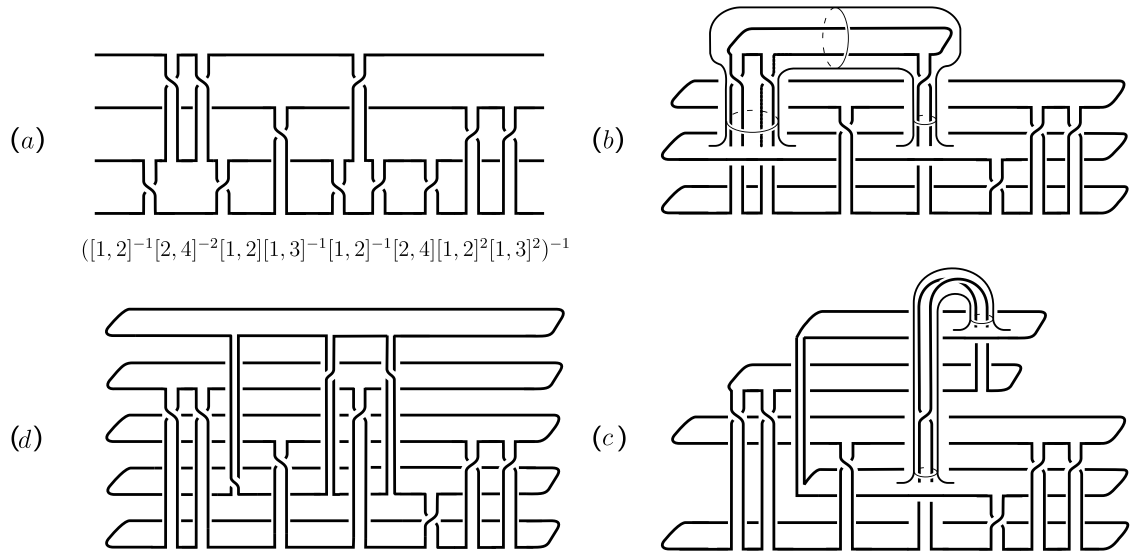


Figure 2:

(a): A genus 4 braided Seifert surface for $16_{1102350}$.
(b): Tube-compression yielding an exotic surface of genus 3.
(c): The left half of the middle disk in (b) is bent and twisted so that the top disk comes out of the tube.
(d): Bennequin surface on a 5-string braid.

We now explain how the examples were found.

For any word

$$e = \prod_{k=1}^{n} [i_k, j_k]^{\varepsilon_k}$$

with $\varepsilon_k = \pm 1$, denote its *crossing number* by

$$c(e) = 2\sum_{k=1}^{n} |i_k - j_k| - n\,,$$

and its *exponent sum* (or *writhe*) by $[e] := \sum_k \varepsilon_k$.

First a list of 11-band knot representations (with a genus 4 surface) admitting a tube compression was generated. We obtained 138,176 words. 1296 of them were discarded as they do not represent prime diagrams. (Such knots are either 3-braids or connected sums of a 3-braid and a 2-braid, and Bennequin's result excludes them.)

Then all genus 3 band representations of $s=4$ strings and $n=9$ bands had to be checked. Since they are too many, this list had to be reduced while being generated. We discarded *a priori* all words with exponent sum $[e]<0$, since the closures of e and e^{-1} are mirrored to each other (and inverted, but orientation is irrelevant here). We also discarded all words with a string connected by only one band, or whose closure is not a knot.

We introduce an order on band words given by crossing number, and for the same crossing number, the words are ordered lexicographically w.r.t. a certain order of the letters. For $[i,j]^{\varepsilon}$ with $1\le i<j\le s$ and $\varepsilon=\pm1$, this order is specified by the lexicographic order of $(j-i,i,\varepsilon)$, that is, for $s=4$ by the increasing sequence of letters

$$[1,2]^{-1}, [1,2], [2,3]^{-1}, [2,3], [3,4]^{-1}, [3,4], [1,3]^{-1}, [1,3], [2,4]^{-1}, [2,4], [1,4]^{-1}, [1,4].$$

Many words can be discarded because they simplify (w.r.t. this order) by band relations. Beside the trivial relations $[i,j]^{\pm1}[i,j]^{\mp1}$, such relations include the commutativity and "band slide" relations. For example if $1\le a<b<c<d\le s$, and e contains a subword $[a,d]^{\pm1}[b,c]^{\pm1}$, or $[c,d]^{\pm1}[a,b]^{\pm1}$ for $d-c\ge a-b$, or $[a,b]^{\pm1}[c,d]^{\pm1}$ for $a-b>c-d$, then a commutativity relation leads to a smaller word, and e can be discarded. When $1\le a<b<c\le s$, band slide relations eliminate all words e with subwords

$$\begin{array}{rll} [a,c]^{\pm1}[a,b], & & \\ [a,c]^{\pm1}[b,c]^{-1}, & & \\ [a,b]^{-1}[a,c]^{\pm1}, & & \\ [b,c][a,c]^{\pm1}, & & \\ [a,c][a,b]^{\pm1} & \text{for} & 2b\ge a+c, \\ [a,b]^{\pm1}[a,c]^{-1} & \text{for} & 2b>a+c, \\ [a,c]^{-1}[b,c]^{\pm1} & \text{for} & 2b\le a+c, \\ [b,c]^{\pm1}[a,c] & \text{for} & 2b<a+c. \end{array}$$

Similarly one can use band slide relations affecting the first and last letter in e. To obtain these relations, for any of the length-2-subwords $\alpha\beta$ on the left of the above 8 families, assume that e is of the form $\alpha\ldots\beta$ (rather than $\ldots\alpha\beta\ldots$). Then interchange the symbol '$\le$' with '$<$', and '$\ge$' with '$>$', in the inequalities describing the conditions for the last 4 families of relations. (That is, the non-strict inequalities become strict and vice versa.)

Then on each word

$$e=\prod_{k=1}^{n}[i_k,j_k]^{\varepsilon_k}\in B_s$$

there is an action of the group $(\mathbb{Z}_s\times\mathbb{Z}_n)\rtimes\mathbb{Z}_2$. Here $1\in\mathbb{Z}_s$ acts by a homomorphism replacing each band $[i,j]$ by $[i+1,j+1] \bmod s$, the residuum $\bmod s$ taken to be between 1 and s, $\mathbb{Z}_n$ acts by cyclic permutations on the letters of e and $\mathbb{Z}_2$ acts by the (anti)involution that takes e into

$$\prod_{k=1}^{n}[s+1-j_{n+1-k},s+1-i_{n+1-k}]^{\varepsilon_{n+1-k}}.$$

We selected only the words which are minimal (w.r.t. our order) in their orbit under this group. In order to make the elimination more efficient, note that the existence of simplifying transformations is already evident from subwords of e. Thus one can build e letter by letter, at each stage checking for such reductions, and discard any initial part of e (and all its potential completions) once such a reduction is found.

This reduced the list to 420,524. Then the set $\mathcal{P}_1$ of skein polynomials of the possible candidates was calculated and compared to the set $\mathcal{P}_2$ of skein polynomials of the genus 3 band words. For the calculation we used the Millett-Ewing program (in the version included in [5]), based on the classical algorithm of Lickorish-Millett [6].

If a possible candidate has a polynomial in $\mathcal{P}_1\setminus\mathcal{P}_2$, then it is an example. In our case $\mathcal{P}_1\setminus\mathcal{P}_2$ was indeed found to be non-empty – 23 of the 665 polynomials in $\mathcal{P}_1$ were not among the 3109 polynomials in $\mathcal{P}_2$. At least three of the examples turned out to have 16 crossings, and thus could be identified in the tables of [5] – they were given above. It was not worth making the above check by identifying first all the knots in the lists (instead of using their polynomials) – because of the large number of diagrams this would have been too time consuming.

To avoid confusion with polynomials of mirror images, we recorded in $\mathcal{P}_1$ and $\mathcal{P}_2$ from each polynomial and its conjugate only the larger one (w.r.t. to some fixed order of the polynomials) so that for finding the examples only the larger one of each pair of (conjugate) polynomials had to be compared.

3. Verifying the examples (proof of Theorem 1)

Even though finding these examples does not require more than an hour of computation, for their check this effort can be significantly reduced.

First, these knots have a skein polynomial of v-span 6 so that by [7, 3] a 4-braid representation must have unique writhe ± 1. (It is 1 for $16_{1102350}$ and 16_{606453} and -1 for 16_{644823} if we use the mirroring convention of (2) rather than that of Figure 1.) By proper mirroring one needs to deal only with writhe 1. This already reduces the list of relevant 9-band representations to 170,472. Using that these representations must not have, or be simplifiable to less than 17 crossings by repeatedly replacing (cyclic) occurrences of $\sigma_i^{\pm 1}\beta\sigma_i^{\mp 1}$ to β, where β has no $\sigma_{i\pm 1}^{\pm 1}$, reduces the list further to 43,980. (We know that our knots have crossing number 16, and any 4-braid with a knot closure has an odd number of crossings.)

The most effective reduction can then be achieved by using that for these knots $\max\deg_z P = 10$ and an estimate of $\max\deg_z P$ in a 4-string band representation (which may be of independent interest). To state this inequality, we need a definition.

Definition 1 Let $e \in B_4$ be a band word with $\mathrm{len}\,(e)$ bands. We call $\mathrm{len}\,(e)$ the *length* of e. Let $p_2(e)$ be the number of (possibly negative) powers of the bands $[1,3]$, $[2,4]$ and $[1,4]$ occurring as (maximal) subwords in e.

That is, if we write

$$e = \prod_{k=1}^{n} [i_k, j_k]^{e_k}\,,$$

with $1 \le i_k < j_k \le 4$, $(i_k, j_k) \ne (i_{k+1}, j_{k+1})$, and $e_k \in \mathbb{Z} \setminus \{0\}$, then $p_2(e) = \#\{\, k : |i_k - j_k| > 1 \,\}$ and $\mathrm{len}\,(e) = \sum_{k=1}^{n} |e_k|$ (whereas the exponent sum $[e] = \sum_{k=1}^{n} e_k$).

Define the *weight* $w(e)$ of e to be

$$w(e) := \mathrm{len}\,(e) + \max(0, p_2(e) - 2)\,.$$

Example 1 For example, all words e in (2) have length $\mathrm{len}\,(e) = 11$, $p_2(e) = 4$ and weight $w(e) = 13$. In the word in (2) for $16_{1102350}$ the 4 maximal subwords counted by $p_2(e)$ are $[2,4]^{-2}$, $[1,3]^{-1}$, $[2,4]$, and $[1,3]^2$.

Lemma 1 If $e \in B_4$ is a band word, then $\max\deg_z P(\hat{e}) \le w(e) - 3$.

Remark 2 This inequality is very often sharp, beside the examples in (2) e.g. also for the infinite series of braids $([1,3][2,4])^k$, $k \in \mathbb{N}$, so that it may not be (easily) improvable.

Turning back to checking our examples, this lemma means that we need to consider only words e of weight $w(e) \ge 13$, and this reduces the list of words to 3051. It did not take more than about 20 seconds to generate and check them, although further reductions are still possible. (For example, the proof of Lemma 1 shows that if e has a subword of the form $[i,i+2]^{\pm 1}[i,i+1]^{\pm 1}[i,i+2]^{\pm 1}$, then we must have $w(e) > 13$.)

Lemma 1 follows by modifying the proof of an inequality in [14, Lemma 1]. For simplicity call the bands $[1,2]$, $[2,3]$ and $[3,4]$ and their inverses *2-bands*, $[1,3]^{\pm 1}$ and $[2,4]^{\pm 1}$ *3-bands*, and $[1,4]^{\pm 1}$ *4-bands*. *Neighbors* will mean the adjacent letters in a band word considered in cyclic order (i.e., first and last letter are also neighbored). The *band crossing* of a band $[i,j]^{\pm 1}$ will be called the crossing corresponding to $\sigma_{j-1}^{\pm 1}$ in (1). In the figures strings will be assumed numbered from left to right and words will be composed from bottom to top.

Proof of Lemma 1. We proceed by induction on the crossing number of the braid and for fixed crossing number on the length of the band-word representation.

If the length of the word $\mathrm{len}(e) \le 3$ the inequality is easy to check. If the word contains only 2-bands, then it follows from [7].

Thus assume $\mathrm{len}(e) > 3$ and that there is at least one 3- or 4-band in e. We apply the skein relation at some band crossing, expressing $P(L_\pm)$ by $P(L_0)$ and $P(L_\mp)$. Since w decreases by 1 or 2 under deletion of a letter, the $P(L_0)$ term is dealt with by induction trivially. Thus we will be concerned how to choose the band (crossing) so that $L_\mp$ can be dealt with by induction.

We can apply the skein relation also at several crossings, thus being allowed to switch several bands before applying induction. This is admissible, because by iterating the skein relation we can write

$$P(L_{\pm,\pm,\dots,\pm}) \;=\; \sum_{\varepsilon_1,\dots,\varepsilon_n\in\{\mp,0\}} \pm z^{f(\varepsilon_1,\dots,\varepsilon_n)}\, v^{g(\varepsilon_1,\dots,\varepsilon_n)}\, P(L_{\varepsilon_1,\dots,\varepsilon_n})\,, \tag{3}$$

with

$$f(\varepsilon_1,\dots,\varepsilon_n) = \#\{\,1\le i\le n:\ \varepsilon_i = 0\,\}\,,$$

and g defined similarly but of no relevance here. Then induction applies automatically to all terms on the right of (3) except for the one with $\varepsilon_1 = \dots = \varepsilon_n = \mp$. We will thus be concerned with choosing the n crossings so that this term can be handled by induction, too.

If e contains some band in a power different from ±1, then we apply the skein relation at one of these letters. Thus assume every band appears in e only in powers ±1.

Assume the word contains two neighbored bands like

$$[i,i+2]^{\pm1}[i+1,i+2]^{\pm1} \text{ or } [i+1,i+2]^{\pm1}[i,i+2]^{\pm1} \text{ or } [i,i+2]^{\pm1}[i,i+1]^{\pm1} \text{ or } [i,i+1]^{\pm1}[i,i+2]^{\pm1}\,, \tag{4}$$

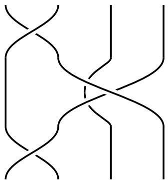
for $i = 1$ or $i = 2$. Then by switching properly their signs, these length-2-subwords simplify to ones of two 2-bands.

Thus assume that subwords as in (4) do not occur in e.

Assume now a 4-band occurs in e. Then eventually neighbored bands $[2,3]$ can be commuted away to one side, so that the 4-band has at least one neighbor different from $[2,3]^{\pm1}$. Then by a similar relation to (4) the 4-band and its neighbor simplify to a 2-band and 3-band. Since the weight does not increase thereby, induction applies.

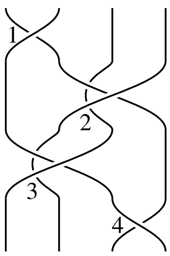
Hence we can assume we have no 4-bands, and there remain the cases where only 2- and 3-bands occur (and at least one 3-band). We distinguish the cases according to the length of a maximal subword of 3-bands occurring in e.

If there is a 3-band with two 2-bands as neighbors (a maximal subword of length 1), then by exclusion of (4), it must look like



Then the two 2-bands can be switched so that one obtains a 4-band (and thus can apply induction because of the shorter band representation).

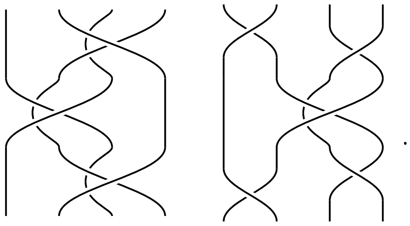
The next situation to consider is the one of a maximal-length-2-subword of 3-bands. Then, up to band switches, the picture looks like



. (5)

We apply now the skein relation at crossing 2. (This, exceptionally, is not a band crossing, so we must argue also about L_0.) Switching crossing 2 annihilates it with crossing 3, and we can group three 2-bands in the lower right part to a 3-band, thus having a situation with the same number of bands, but one 3-band and 2 crossings less than in (5). Nullifying crossing 2 makes crossings 1 and 4 annihilate, thus giving a 3-band and two 2-bands, and thus a weight at least by one less than in (5), so induction applies here, too.

If there is a 3-band with two 3-band neighbors (a maximal subword of length at least 3), then up to band switches it will look like picture (a) in



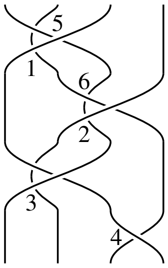
(a) (b)

Assume these three 3-bands are not the only ones in e. The braid word in picture (a) turns into the one in picture (b). Latter braid word has two crossings less, two more bands (four 2-bands and one 3-band), but two 3-bands less than picture (a), so that induction applies. (This is where the extra contribution from the 3-bands in the definition of w is needed.)

There is the exceptional, and final, situation where this argument cannot be applied, namely when these three bands are the only 3-bands in e. In this case picture (b) has higher weight.

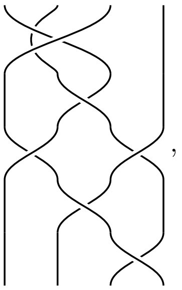
If e has only three 3-bands forming a subword, then there must be at least one 2-band in e, otherwise after a cyclic letter permutation we can obtain a word of smaller weight. Then (after the allowed band crossing switches) the picture looks like in (5), but with the uppermost band being a 3-band.

$L_\pm \quad =$



. (6)

We deal with it similarly, applying the skein relation at crossing 2. $L_\mp$ looks like



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and can be written as two 3-bands and two 2-bands. In L_0 after crossings 1 and 4 also crossings 5 and 6 cancel, so we have one 3-band and two 2-bands, and can apply induction. □

4. Some problems

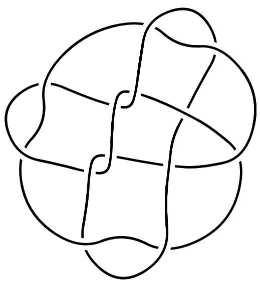
The use of the skein polynomial had the advantage of making verifiable the inequality $\min\deg_v P \leq 2g$ conjectured by Morton [8]. Indeed, this inequality was satisfied for all the knots we could decide (in particular, for all knots with Bennequin surfaces of 4-braid representations with at most nine bands). However, in genus 4 the second author subsequently found counterexamples to Morton's inequality [15].

In the simpler cases of knots of genus 1 and 2 we were unable to find examples for Theorem 1. One can investigate for other examples of similar type, in particular when specializing band representations from being of minimal genus to being positive (which are of minimal genus by Bennequin's inequality [1, theorem 3]).

Question 1 Are there knots with positive band representations, but no such representations of minimal string number?

In view of the examples given here, and those for positive (crossing) representations in [15], one may expect a positive answer. In connection with this, we conclude with the following related example, which deserves independent mention.

Example 2 The knot 13_{9684} (Figure 3) is positive (and hence has a positive band representation [10, 12]) of genus 3, and its skein polynomial does not occur in the above computed set $\mathcal{P}_2$. Thus 13_{9684} has no 4-braid representation with nine bands, in particular no positive one. Its skein polynomial has v-span 6, admitting the possibility that it has a representation as a 4-braid of writhe 9. This possibility is also compatible with the v-degrees of the skein polynomial of the 2-cable. (In particular, the v-span is 14, and the trick applied in [15] to 14_{45759} does not work.) On the other hand, we could not find a 4-braid representation of this knot with up to 19 crossings. Since the check of 21 crossing representations, and applying of Morton-Williams-Franks to the skein polynomial of a 3-cable link, which has (in its obvious diagrams) at least 117 crossings, does not seem computationally reasonable, we are unable to decide whether 13_{9684} is a knot of the type we seek or not. Certainly, however, this example shows the serious difficulties in determining the braid index, despite the effectivity of the Morton-Williams-Franks inequality (and the few alternative methods amplifying it) for the few common knots of small crossing number.



13_{9684}

Figure 3: Does this knot have braid index 4?

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