GRID DIAGRAMS, LINK INDICES, AND THE HOMFLY-PT POLYNOMIAL

Hwa Jeong Lee(†), Alexander Stoimenov

Dongguk University, WISE campus,
Department of Mathematics Education,
123, Dongdae-ro,
38066 Gyeongju-si, Republic of Korea
e-mail: stoimeno@stoimenov.net,
WWW: http://stoimenov.net/stoimeno/homepage/

Gyo Taek Jin

Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology,
34141 Daejeon, Republic of Korea
e-mail: trefoil@kaist.ac.kr,

August 10, 2025

Abstract. Continuing our study of Euler characteristic 0 braided surfaces as grid diagrams, we employ the HOMFLY-PT polynomial to some rather sharp ("frying eggs in the pan"-style) estimates of Thurston-Bennequin invariants and arc index of knots. We also relate this to strong quasipositivity of Whitehead doubles. We can resolve the braid index and the existence of minimal string Bennequin/strongly quasipositive surface for Whitehead doubles of alternating knots, among others.

Keywords: arc index, braid index, link polynomial, Thurston-Bennequin invariant, quasipositive, strongly quasipositive, Whitehead double.

2020 AMS subject classification: 57K10 (primary), 57M50, 20F36, 57K14, 57K33 (secondary)

1 Introduction

This is the second part of a long account on an investigation resulting from attempts to understand braided surfaces, in particular Bennequin and strongly quasipositive surfaces. As it turned out, even in the simplest case of Euler characteristic 0, the answer is revealingly complicated, in that these surfaces are essentially equivalent to grid diagrams D for knots. However, these grid diagrams D are also equipped with a framing $\lambda(D)$, and in the first part [JLS] many consequences were discussed of the identification of this framing with the (negated) Thurston-Bennequin invariant (Theorem 3.6).

Building on that study, we treat here the HOMFLY-PT polynomial P. The arc index has the Morton-Beltrami lower bound (5.11) [MB] which, by the work of Dynnikov-Prasolov [DP] (Theorem 4.10), is refined by the Kauffman polynomial bound for the Thurston-Bennequin invariant (3.15). Our main contribution here is that there is an alternative pair of inequalities (1.2,5.73) to (5.11,3.15) using the HOMFLY-PT polynomial P. These new estimates are provably better in a variety of cases (see Corollariy 5.19 and

Proposition 5.43, and Computation 5.44). This includes optimality for alternating knots (Proposition 5.17) and positive knots K (Corollary 5.45). These bounds have their own new geometric applications.

An outline of the paper is as follows.

After compiling preliminaries in §2, we recall in §3 previous work in [JLS]. We review that Euler characteristic 0 braided surfaces are essentially grid diagrams D, with a framing attached, which we write as $\lambda(D)$. When the surface is strongly quasipositive, then

$$\lambda(D) = -TB(D) \tag{1.1}$$

was identified, up to sign, with the Thurston-Bennequin invariant of D. As we explain, in conformance with (1.1), we will usually write $\lambda(K) = -TB(K)$.

In §4 we discuss the braid index $b(K)$ and its variants for Bennequin and strongly quasipositive surfaces, and how the arc index $a(K)$ is fundamentally connected to a braid index $b(A(K,t))$ (see Corollary 4.1 and Conjecture 6.1). We also introduce the framing diagram $\Phi(K)$ of a knot K (Definition 3.9) and its cone structure (Definition 4.7).

After these preparations, we move to the main work of this paper in §5, which is a detailed treatment of the HOMFLY-PT polynomial. The possibility exists (Conjecture 2.3) that the HOMFLY-PT polynomial determines the braid index, thus this could be true for the arc index as well. In the simplest form, we extract (in a "culinary" way) an invariant, we call $l(K)$, which gives a lower bound for the arc index of K,

$$l(K) \le a(K) \tag{1.2}$$

(see Theorem 5.7). It (apparently, see Question 5.14) already improves upon the Morton-Beltrami [MB] bound.

For (even) better estimates, one can use cabling, and to limit complexity problems, we introduce partial cabling (Lemma 5.25). This can be complemented by some extra arguments, and shows that the HOMFLY-PT polynomial is efficient to practically determine the arc index (see Lemma 5.13 and Remark 5.28) and maximal Thurston-Bennequin number (Proposition 5.33) in most examples. We further outline (end of §5.3) how to apply the Kauffman polynomial beyond the Morton-Beltrami inequality, and also prove the Finite-Cone-Theorem 5.3.

Section §6 mostly deals with a summary of previous considerations, including more explicit forms of the Finite-Cone-Theorem (Propositions 6.6 and 6.7). We also highlight potential pathologies about non-coincidence of various types of braid indices. This comprises Rudolph's problem (4.1). We show that the l-invariant can be also used to exclude such odd behavior (Propositions 6.9 and 6.11). This leads to an extension of the result of Diao and Morton [DM] (Proposition 6.14). Among further applications is the following.

Corollary 6.13′ Assume K is alternating and $L = W_\pm(K,t)$ or $L = A(K,t)$ for some t. Then L has a minimal string Bennequin surface. Also, if L is strongly quasipositive, then L has a minimal string strongly quasipositive band presentation.

Throughout the treatise, we encounter many suggestive but difficult to resolve questions. We have deliberately put emphasis on them, since their examination would provide various directions to deepen the present consideration.

The third (and final) part of this sequence of papers [St] is written by the second author and discusses what previous results on strong quasipositivity can be extended to quasipositivity. When strong quasipositivity is replaced by quasipositivity, then many considerations revolve around sliceness. This is closely related to the problem of slicing Whitehead doubles, and we will extra need both Casson-Gordon and Vassiliev invariants.

2 Definitions and Preliminaries

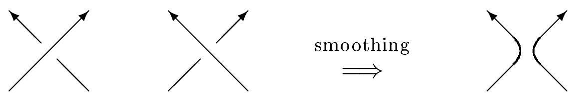
Many of the definitions and notations needed coincide with those given in [JLS], but must inevitably be included again, in order to make this paper self-contained.

2.1 Generalities

We say an inequality '$a \geq b$' is *sharp* or *exact* if $a = b$ and *strict* (or *unsharp*) if $a > b$. We use $\#E$ for the cardinality of a finite set E and $\lfloor x \rfloor$ for 'greatest integer' part of $x \in \mathbb{Q}$.

2.2 Links and genera

All link diagrams and links are assumed oriented. Crossings in an oriented diagram D of a knot K are called as follows.



positive negative smoothed out

(2.1)

The *sign* of a positive/negative crossing is assigned to be ± 1 accordingly. Let $c_\pm(D)$ be the number of positive, respectively negative crossings of a link diagram D, so that the *crossing number* of D is $c(D) = c_+(D) + c_-(D)$ and its *writhe* is $w(D) = c_+(D) - c_-(D)$. We write $s(D)$ for the *number of Seifert circles* of D, which are the circles obtained after smoothing all crossings of D. We write $c(K)$ for the crossing number of a knot K, the minimal crossing number of all diagrams of K. The mirror image of K will be written $!K$, and the mirror image of diagram D (in the form obtained by switching all crossings of D) will be $!D$. If $K = !K$ (up to orientation), we call K *amphicheiral*. We use '$\bigcirc$' to denote the *unknot* (trivial knot) in formulas, and $T_{p,q}$ is used for the (p,q)-*torus knot*.

The symbol '#' is used for *connected sum* (as a binary operation on links, unlike its previous introduction as 'cardinality'). The number of components of a link L is denoted by $\kappa(L)$. The *bridge number* $br(L)$ of L is the minimal number of Morse maxima of L (or equivalently, of any diagram of L). The *(Seifert) genus* $g(L)$ resp. *Euler characteristic* $\chi(L)$ of a knot or link L is said to be the minimal genus resp. maximal Euler characteristic of a *Seifert surface* of L. We have

$$2g(L) = 2 - \kappa(L) - \chi(L)\,.$$

Similarly write $\chi_4(L)$ for the *smooth 4-ball* (maximal) Euler characteristic and

$$2g_4(L) = 2 - \kappa(L) - \chi_4(L)\,.$$

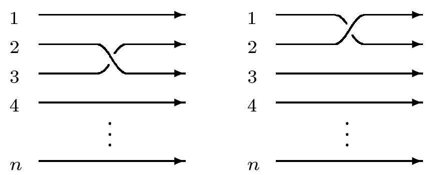
(In the following 4-ball genera and sliceness will always be understood smoothly.) A knot K is *slice* if $g_4(K) = 0$, or equivalently, $\chi_4(K) = 1$. We will refer to the following basic fact: if $\kappa(L) = 2$ and $\chi_4(L) = 2$, then both components of L must be slice (knots), and have linking number 0.

2.3 Braids and braided surfaces

We write B_n for the *braid group* on n *strands* or *strings*. The relations between the *Artin generators* σ_i, $i = 1, \dots, n-1$ are given by

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$ and
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $1 \leq i < j - 1 \leq n-2$.

In diagrams we will orient braids left to right and number strings from top to bottom, for example:



σ_2 σ_1^{-1}

There is a *permutation homomorphism* $\pi : B_n \to S_n$, sending each σ_i to the transposition of i and $i+1$. By a *subbraid* of $\beta \in B_n$ we mean a braid obtained by taking only a subset $C \subset \{1,\dots,n\}$ of the strands in β, which is invariant under the associated permutation $\pi(\beta)$ of β (i.e., C is a union of cycles of $\pi(\beta)$).

We define *band generators* in B_n by

$$\sigma_{i,j} = \sigma_i \dots \sigma_{j-2}\sigma_{j-1}\sigma_{j-2}^{-1} \dots \sigma_i^{-1}\,, \tag{2.2}$$

Notice that $\sigma_{i,i+1} = \sigma_i$. A representation of a braid $\beta \in B_n$ in the form

$$\beta = \prod_{k=1}^{l} \sigma_{i_k,j_k}^{\pm 1} \tag{2.3}$$

is called a *band presentation*. (See e.g. [BKL].) Usually, it will be more legible to use the symbol

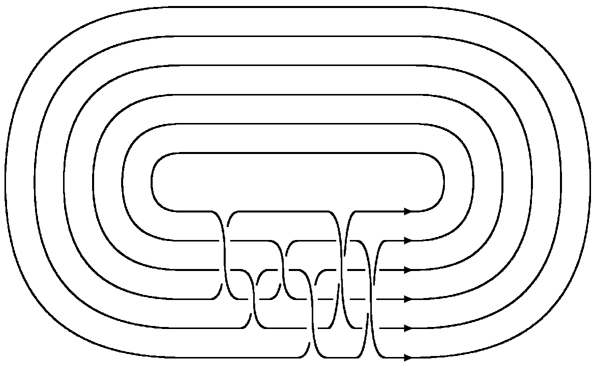
$$[ij] = \sigma_{i,j}$$

when writing band generators in formulas. Similarly we use $-[ij] = \sigma_{i,j}^{-1}$. In certain cases, we even omit the brackets (see Definition 3.4 and Example 5.27). Also, when $j = i+1$, we often simply write i for σ_i and $-i$ for σ_i^{-1}, when no ambiguity arises.

The image of β under the abelianization $B_n \to \mathbb{Z}$ is the *writhe* (or exponent sum) of β, and is written $w(\beta)$. This quantity can be calculated from the exponent sum on the right of (2.3).

In Definition 3.4 we will extend suitable words in $[ij]$, without negative exponents, also to encode grid diagrams.

A braid $\beta \in B_n$ whose *closure* $\hat{\beta}$ is the link L is a *braid representative* of L. Similarly a word for β gives a (braid closure) diagram $D = \hat{\beta}$ of L. When β is a word, then $w(\hat{\beta}) = w(\beta)$. A band presentation β naturally spans a Seifert surface of $L = \hat{\beta}$. Following Rudolph, we call this a *braided surface* of L. For example, $n = 6$ and $l = 6$,



for the 6-braid $\beta = \sigma_{1,4}\sigma_{3,5}\sigma_{2,4}\sigma_{3,6}\sigma_{1,5}\sigma_{2,6}$. The diagram shows the closure $L = \hat{\beta}$. It is easily seen that the six 'elliptic' disks joined two by two with six twisted bands form a natural Seifert surface of L. Rudolph [Ru] proves that every Seifert surface is a braided surface. If a braided surface is of minimal genus for L, it is called a *Bennequin surface* of L [BM2].

A link is called *quasipositive* if it is the closure of a braid β of the form

$$\beta = \prod_{k=1}^{\mu} w_k \sigma_{i_k} w_k^{-1} \tag{2.4}$$

where w_k is any braid word and σ_{i_k} is a (positive) standard Artin generator of the braid group. (In [Ru4] there is some overview of this topic.) If the words $w_k \sigma_{i_k} w_k^{-1}$ are of the form σ_{i_k,j_k} in (2.2), so that

$$\beta = \prod_{k=1}^{\mu} \sigma_{i_k,j_k}\,, \tag{2.5}$$

then they can be regarded as embedded bands. Links which arise this way, i.e., such with *positive band presentations*, are called *strongly quasipositive links*.

Bennequin's inequality [Be, Theorem 3] states

$$-\chi(L) \geq w - n \tag{2.6}$$

for an n-strand braid representative of L of writhe w. If there is a braid representative β of L making (2.6) an equality, we call both L and β *Bennequin-sharp*. This inequality was later extended to

$$-\chi(L) \geq -\chi_4(L) \geq w - n \tag{2.7}$$

(see e.g. [IS, St2]). In an analogous way we defined that L and β are *slice-Bennequin-sharp*.

It implies that a *strongly quasipositive surface*, i.e., obtained from a positive band presentation, is minimal genus. Namely, a positive band presentation of w bands on n braid strands gives a braid of writhe w. Thus the surface S constructed from the band presentation yields, with (2.7),

$$-\chi(L) \leq -\chi(S) = w - n \leq -\chi_4(L) \leq -\chi(L)\,.$$

This also shows that a strongly quasipositive link L is always Bennequin-sharp, and

$$\chi_4(L) = \chi(L)\,. \tag{2.8}$$

The *Bennequin sharpness conjecture* (see [FLL, St2]) asserts

$$L \text{ is Bennequin-sharp} \iff L \text{ is strongly quasipositive}\,. \tag{2.9}$$

For some related results, see [JLS, St].

Definition 2.1
- Let $b(K)$ be the braid index of K, the minimal number of strings of a braid representative of K.
- Let $b_b(K)$ be the Bennequin braid index of K, the minimal number of strings to span a Bennequin surface of K.
- When K is strongly quasipositive, let $b_{sqp}(K)$ be the minimal number of strings to span a strongly quasipositive surface of K (only positive bands).
- Further, for a Seifert surface S, let $b(S)$ be the minimal string number on which S is spanned as a braided surface.
- If S is a strongly quasipositive surface, let $b_{sqp}(S)$ be the minimal string number on which S is spanned as such (i.e., arises from a positive band presentation).

We have then (with the right inequality only valid for strongly quasipositive K)

$$b(K) \leq b_b(K) \leq b_{sqp}(K)\,, \tag{2.10}$$

and by definition, with S being a Seifert surface of K,

$$b_b(K) = \min\{\, b(S) \,:\, \chi(S) = \chi(K)\,\}\,, \qquad b_{sqp}(K) = \min\{\, b_{sqp}(S) \,:\, S \text{ strongly quasipositive}\,\}\,. \tag{2.11}$$

We will further discuss these relations in §4 and §6. We also feature the following result. It confirms an expectation originally formulated for $n = b(L)$ by Jones [J, end of §8] (later also referred to as the "weak" form) and subsequently extended by Kawamuro.

Theorem 2.2 (proof of the Jones-Kawamuro conjecture [DP, LaM]) For every link L, there is a number $w_{min}(L)$, so that every braid representative β of L on n strands of writhe w satisfies

$$|w - w_{min}(L)| \leq n - b(L)\,. \tag{2.12}$$

Generally speaking, we will use this theorem to advance theoretical applications in our work, but for practical ones, another tool will be crucial, which we introduce next.

2.4 Link polynomials

We use the *HOMFLY-PT polynomial* P [LiM], in the Morton [Mo] convention

$$P(\bigcirc) = 1\,, \qquad v^{-1}P_+ - vP_- = zP_0\,, \tag{2.13}$$

where P_+, P_- and P_0 refer to the polynomials of three links with diagrams equal except at one spot, where they contain the fragments of (2.1) from left to right. The right part of (2.13) is also called P's *skein relation*. We will use the suggestive notation $\min\deg_v P$ for minimal v-degree of (any monomial in) P, and similarly $\max\deg_v P$, and set $\operatorname{span}_v P = \max\deg_v P - \min\deg_v P$. We write $[P]_{z^k}$ for the coefficient of z^k in P, being a polynomial in v. Then, $[P]_{v^d}$ the coefficient of degree d in v (which is itself treated as a polynomial in z). Also set

$$\min\operatorname{cf}_v P = [P]_{v^{\min\deg_v P}} \tag{2.14}$$

to be the trailing (lowest degree) coefficient of P. The notation $P|_{z^{\ge k}}$, $P|_{z^{\le k}}$, and $P|_{z^{\ne k}}$ will mean (the polynomial consisting of) all terms in P of z-degree at least k, at most k, and different from k, respectively. The z-variable is left inside. Thus $[P]_{z^k}$ is a polynomial in v, while $P|_{z^{\ge k}}$ is a polynomial in z, v. We occasionally refer to $P|_{z^{\le k}}$ as a *(z-)truncated polynomial*. We emphasize that much of the useful information of P can be obtained from truncations thereof (like (2.20)), which are much faster (subexponentially) to compute than the full polynomial. A program that calculates such truncations was introduced in [St3], and we will extensively apply it below.

A CPU-parallelized upgrade of the truncated polynomial calculation was developed to settle the last 16 crossing prime knot standing to resolve for the below question (4.1); it has now its own description page on [St4].

Two further standard properties of P are that for a link L of $\kappa(L)$ components, $\min\deg_z P(L) = 1-\kappa(L)$, and $P(L)$ contains only monomials $z^p v^q$ for p, q odd (resp. even) when $\kappa(L)$ is even (resp. odd). The mirroring behavior of P is (signed) v-conjugation:

$$P(!L)(v,z) = (-1)^{\kappa(L)-1}P(L)(v^{-1},z)\,. \tag{2.15}$$

We further use the identity (see [LiM, Proposition 21])

$$P(v, v^{-1}-v) = 1\,. \tag{2.16}$$

By the MFW [Mo, FW] inequalities, the writhe w of an n-string band presentation of L satisfies

$$w+n-1 \ge \max\deg_v P(L) \ge \min\deg_v P(L) \ge w-n+1\,, \tag{2.17}$$

thus

$$\mathrm{MFW}(L) := \frac{1}{2}\operatorname{span}_v P(L) + 1 \le b(L)\,, \tag{2.18}$$

where the left hand-side is the *MFW bound* for the braid index $b(L)$. If $\mathrm{MFW}(L) = b(L)$, we call L *MFW-sharp*.

When L is not MFW-sharp, there are ways to improve the braid index estimate using cables of L: when L' is a degree-c cable of L, then

$$\mathrm{MFW}(L') \le b(L') \le cb(L)\,,$$

thus

$$b(L) \ge \left\lceil \frac{1}{c}\mathrm{MFW}(L') \right\rceil\,. \tag{2.19}$$

The method is well explained in [MS] (certainly when $c=2$; some examples for $c=3,4$ can be found in [St3]). We refer to such estimates as the *cabled MFW*.

To relate this to the Jones-Kawamuro conjecture (Theorem 2.2), we point out that MFW plus cabled versions thereof is efficient to determine the braid index of most links. In some cases alternative methods apply, but for every link L whose braid index is decided so far, (2.19) is known give a sharp estimate at least for sufficiently large c. It is thus conjecturable that this is always the case (see [St4]):

Conjecture 2.3 For every link L there is a $c>0$ and a degree-c cable link L' of L making (2.19) sharp.

Obviously, when we can prove that a braid representative β of a link L is minimal, then we immediately also obtain $w_{min}(L)=w(\beta)$ in Theorem 2.2. However, it was also noticed in [St5] that once (2.19) (for some c) gives a sharp estimate of $b(L)$, it proves along the way that $w_{min}(L)=w(\beta)$ is unique. (And it is not too hard to derive (2.12) either from that argument.) Thus Theorem 2.2 provides a theoretical underpinning, but is neither very practically helpful nor essential to determine $b(L)$ or $w_{min}(L)$ for a given L.

One main drawback of (2.19) is that in general the polynomial of a cable link L' is notoriously hard to calculate. But instead of the whole polynomial, we can use a truncation:

$$\mathrm{MFW}_d(L')=\frac{1}{2}\,\mathrm{span}_v P(L')|_{z\le d}+1\le \mathrm{MFW}(L')\le b(L')\,. \tag{2.20}$$

We refer below to such type of estimate of the braid index as *truncated (cabled) MFW*.

When quoting specific computations of P polynomials (see e.g. Table 2 or Example 5.42), the notation should be read thus. The first line contains the crossing number of the diagram the polynomial was computed from, an identifier, and $\min\deg_z P$ and $\max\deg_z P$. Then in each line follow $[P]_{z^d}$ for $\min\deg_z P\le d\le \max\deg_z P$ with $d-\min\deg_z P$ even. The line starts with $\min\deg_v[P]_{z^d}$, then $\max\deg_v[P]_{z^d}$, and then follow the coefficients $[P]_{z^dv^e}$ with $\min\deg_v[P]_{z^d}\le e\le \max\deg_v[P]_{z^d}$ and $e-\min\deg_v[P]_{z^d}$ even. These entries are aligned so that coefficients in the same v-degree are on the same left-right position.

Returning to surfaces, it follows from the right inequality in (2.17) that a Bennequin-sharp (in particular strongly quasipositive) link L satisfies

$$\min\deg_v P(L)\ge 1-\chi(L)\,. \tag{2.21}$$

Morton also proves in [Mo] the *canonical genus inequality*, for any diagram D of L,

$$\max\deg_z P(L)\le c(D)-s(D)+1\,. \tag{2.22}$$

The *Conway polynomial* ∇ is given by

$$\nabla(L)(z)=P(L)(1,z)\,. \tag{2.23}$$

The *determinant* of a knot K can be defined by

$$\det(K)=|\nabla(2\sqrt{-1})|\,. \tag{2.24}$$

This is always an odd number (when K is a knot).

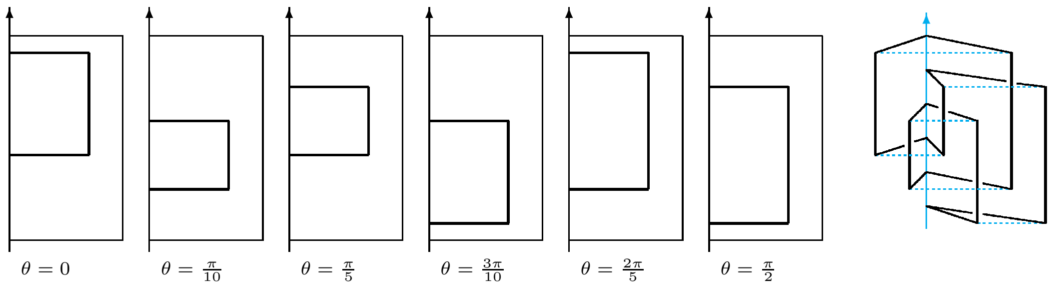
The *Kauffman polynomial* $F=F(a,z)(K)$ will be needed at a few places for reference. We use the following well-known properties: for every link L,

- $F(L)$ contains only monomials a^pz^q for $p+q$ even.
- $$F(\sqrt{-1},z)(L)=1\,. \tag{2.25}$$
- For a knot K,
  $$[F(K)]_{z^0}(\sqrt{-1}v)=[P(K)]_{z^0}(v)\,, \tag{2.26}$$
  and
- the Kauffman-Jones substitution
  $$F(-t^{3/4},t^{1/4}+t^{-1/4})(L)=V(L) \tag{2.27}$$

We caution that our mirroring convention is so that the positive (right-hand) trefoil 3_1 has $\min\deg_a F(3_1)=1$ and $\max\deg_a F(3_1)=4$. (This convention is, e.g., opposite to [DM, Th], i.e., with a and a^{-1} interchanged.)

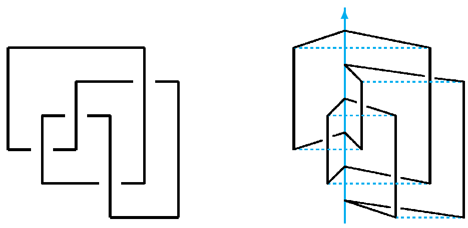
2.5 Grid diagrams and arc index

An *arc presentation* of a knot or a link L is an ambient isotopic image of L contained in the union of finitely many half planes, called *pages*, with a common boundary line in such a way that each half plane contains a properly embedded single arc.



A *grid diagram* (or, for simplicity simply called *grid* often below) is a knot or link diagram which is composed of finitely many horizontal edges and the same number of vertical edges such that vertical edges always cross over horizontal edges. We assume that horizontal/vertical positions of vertical/horizontal edges are pairwise distinct. In particular, away from crossings edges only meet at corners, and vertices are pairwise distinct.

It is not hard to see that every knot admits a grid diagram (compare with (3.17)). The figure below explains that every knot admits an arc presentation.



We set the *size* $\mu(D)$ of a grid diagram to be the number of vertical *or* (equivalently) horizontal segments (but *not both* together). A grid (diagram) of size μ will also be shortly called a μ-*grid*.

In general, we will afford the sloppiness of abolishing the distinction between an ordinary and a grid diagram, whenever the grid structure is unnecessary. Thus, for instance, $c(D)$ can mean the crossing number of both an ordinary and grid diagram, whereas $\mu(D)$ would imperatively assume that D is given a grid shape.

Let $a(L)$ be the *arc index* of L, the minimal $\mu(D)$ over all grid diagrams D of L. It is the minimal number of pages among all arc presentations of a link L.

We note that the following was proved by Cromwell [Cr]. For two links L_1, L_2,

$$a(L_1 \# L_2) = a(L_1) + a(L_2) - 2\,. \tag{2.28}$$

For knots L_i, it also follows from a relationship (5.68), derived by Dynnikov-Prasolov [DP], concerning the Thurston–Bennequin invariant (see §3 for notation), and the additivity of the invariant [EH, To].

2.6 Knot tables

For notation from knot tables, we follow Rolfsen's [Ro, Appendix] numbering up to 10 crossings, except for the removal of the Perko duplication.

For 11 to 16 crossings we use the tables of [HT] (which for 11 to 13 crossing knots are now also on KnotInfo [LvM]), while appending non-alternating knots after alternating ones of the same crossing number. Thus, for instance, $11a[k] = 11_{[k]}$ for $1 \le [k] \le 367$, and $12n[k] = 12_{1288+[k]}$ for $1 \le [k] \le 888$.

For non-alternating knots of 17 and 18 crossings (end of §5.2.2), we used Burton's census, [Bu]. (It includes, but again reorders, the pre-existing tables up to 16 crossings.)

If it is relevant, mirror images will be distinguished on a case-by-case basis. Specifically, for the $(2, n)$-torus knots, we will say that the knot is *positively/negatively mirrored*. The convention for 10_{132} is fixed in Example 3.7. (The knot exhibits certain phenomena that have to be treated for higher crossing knots as well, but being the only Rolfsen knot with such status, it will merit detailed attention.)

3 Thurston-Bennequin invariant

3.1 Weight model for the Thurston-Bennequin invariant

The main topic of the work in [JLS] started from the observation (probably first occurred to Nutt [Nu]) that a braided surface of Euler characteristic 0, which is a K-knotted annulus, is essentially a grid diagram of the underlying companion knot K. In this section we review definitions and results (mostly without repeating proofs) from [JLS].

Definition 3.1 Let for a knot K and integer t,

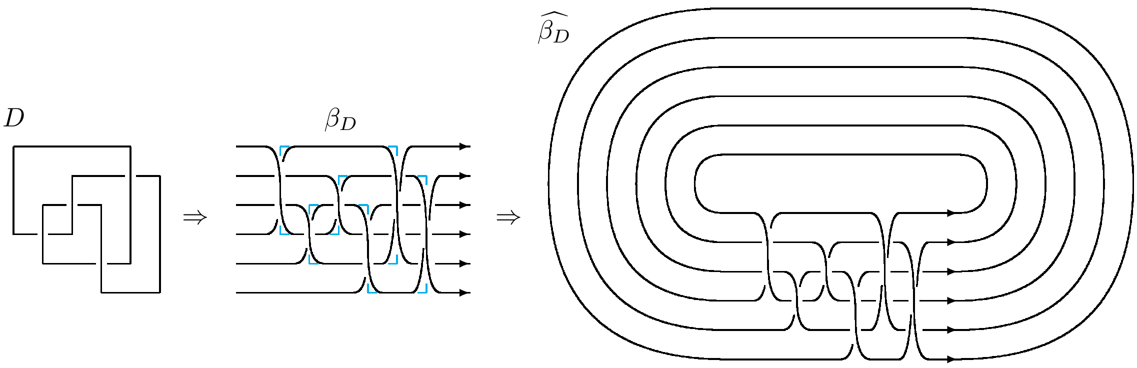
- $A(K, t)$ be the (link of the) t-framed K-knotted annulus,
- $W_+(K, t)$ and $W_-(K, t)$ the t-framed Whitehead doubles of K with positive and negative clasp, and
- $B(K, t)$ the t-framed Bing double of K.

We will usually abuse the distinction between the annulus and the link which is its boundary.

To disambiguate among different conventions for framing used elsewhere, we emphasize that t is here the linking number of the two components of $A(K, t)$. Thus, for example, $A(\bigcirc, 1)$ is the positive (right-hand) Hopf link, and $A(\bigcirc, -1)$ the negative one. This definition of framing has the opposite sign to the one used by other authors (e.g., [DM]), where they take the writhe $w(D) = -t$ of a diagram D of K from which $A(K, t)$ is constructed as the blackboard-framed (reverse) 2-parallel.

Also, $W_+(\bigcirc, 1)$ is the positive (right-hand) trefoil, and $W_+(\bigcirc, -1) = W_-(\bigcirc, 1)$ the figure-8-knot. We can understand $W_+(K, t)$ resp. $W_-(K, t)$ as the result of plumbing a positive resp. negative Hopf band into $A(K, t)$ and taking the knot which is the boundary of the resulting Seifert surface. In a similar way, we can understand $B(K, t)$ as the 2-component link which is obtained by plumbing both a positive and a negative Hopf band into $A(K, t)$ and taking the boundary. Thus for instance $B(\bigcirc, 0)$ is the 2-component unlink, and $B(\bigcirc, 1)$ is the Whitehead link.

Let D be a grid diagram of a knot K. Replacing each vertical segment with a half twisted band as shown below, we get a braid in band presentation, denoted by β_D. (Compare with [Nu, Theorem 3.1].) Then the closure $\widehat{\beta_D}$ bounds a twisted annulus. Therefore $\widehat{\beta_D} = A(K, t)$ for some t.



(3.1)

Consider the situation that the band presentation is positive. Then obviously $A(K,t)$ for the resulting framing t is strongly quasipositive. A question is what is the framing t, which we will write as

$$t = \lambda(D)\,, \tag{3.2}$$

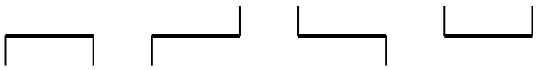
in dependence of the diagram D, and how to read $\lambda(D)$ off D. To explain the formula for $\lambda(D)$, given below as (3.5), we fix some notation.

Let the *weight* of a grid diagram D be

$$Z(D) = \frac{1}{2} \sum_{e \text{ edge of } D} \operatorname{sgn}(e)\,, \tag{3.3}$$

where the signs of the edges are determined as follows:

$$\operatorname{sgn}(e) = \begin{cases} 1 & e \text{ is vertical} \\ \pm 1,\, 0 & e \text{ is horizontal and one of the following forms} \end{cases}$$



0   +1   −1   0 (3.4)

Remark 3.2 This weight formula (3.3) can be generalized to non-positive band presentations by letting each vertical edge have the sign of the corresponding band. But we will treat this more general case only occasionally here.

Lemma 3.3 With $w(D)$ being the writhe, we have

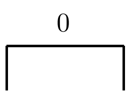
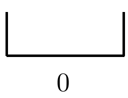
$$\lambda(D) = Z(D) - w(D)\,. \tag{3.5}$$

Definition 3.4 Also, we can use the band presentation of β_D to specify the grid diagram D itself (see Example 5.27). The mirroring of D is fixed by default by saying that β_D should be obtained when reading D from the left. This means that we can write the grid diagram D in (3.1), even omitting brackets, as

`14 35 24 36 15 26.`

Since we deal with grids of size 10 or more, let us also already fix here that we use initial capital Latin letters `A`,`B`,`C`,... to denote two-digit integers $10, 11, 12, \ldots$, so that for example, `4C` $= [4, 12] = \sigma_{4,12}$.

Let $br(D)$ be the *vertical bridge number* of D, which is the number of sign-0 horizontal edges of D of one of either types in (3.4)

$br(D) := \#\Big($  $\Big) = \#\Big($  $\Big)$ (3.6)

Definition 3.5 We set $\lambda_{min}(K) = \lambda(D)$ whenever $\mu(D) = a(K)$.

We will use $\lambda_{min}(K)$ often in the following. Two caveats are in order regarding this notation. First, the 'min' refers to the minimum with respect to number of strings of the surface $A(K,t)$ (or horizontal segments in the grid diagram of K), *not* the framing t itself. And second, it is *not* assumed that λ_{min} is unique. At least for the unknot K,

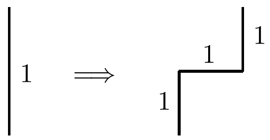
$$\text{both } b(A(\bigcirc, 0)) = b(A(\bigcirc, 1)) = 2, \text{ thus } \lambda_{min}(\bigcirc) = 0, 1. \tag{3.7}$$

This special behavior of unknot will require repeated attention. For a non-trivial knot K, the uniqueness and minimality of $\lambda_{min}(K)$ was settled, as will be discussed below; see Theorem 4.10. But we do not wish to exclude $K = \bigcirc$ consistently. We prefer to maintain the symbol $\lambda_{min}(K)$, stipulating that formulas involving $\lambda_{min}(K)$ are meant to hold whatever of either values (3.7) is chosen for $K = \bigcirc$. For $K \neq \bigcirc$, the reader may assume that

$$\lambda_{min}(K) = \lambda(K)\,, \tag{3.8}$$

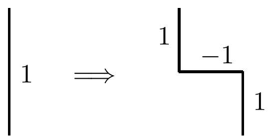
though we will not use this before stating Theorem 4.10.

We also observed that when μ is augmented by 1, we can always augment by 1,



(3.9)

resp. preserve

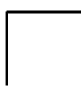
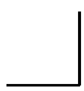


(3.10)

any given framing $\lambda(D)$ by the above two moves. We call these moves in the following *positive* and *negative stabilization*, resp. Thus, $\lambda(D)$ augments by 1 under positive stabilization, and negative stabilization does not change $\lambda(D)$.

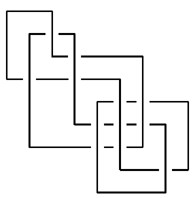
The *Thurston-Bennequin invariant* $TB(D)$ of a grid diagram D can be defined as is being identified in the following theorem.

Theorem 3.6 For any grid diagram D, the quantity $Z(D)$ counts the NW- or SE-corners of D.

$Z(D) = \#\Big($  NW-corners$\Big) = \#\Big($  SE-corners$\Big)$ (3.11)

Thus, by Lemma 3.3, (1.1) holds.

Example 3.7 The [J+] diagram D of 10_{132},



10_{132}

read from the left, gives the 9-strand band presentation

$$\beta_D = [14][27][13][26][59][48][37][69][58]\,. \tag{3.12}$$

We have $\mu(D) = 9$, $Z(D) = 3$, $w(D) = 2$, $br(D) = 3$ and $\lambda(D) = 1$. Thus (3.12) gives a (positive) band presentation of $A(10_{132}, 1)$. The mirroring of 10_{132}, determined by D, is so that it *has the P polynomial of the positively mirrored* 5_1. We fix this mirroring in the sequel, since we will illustratively feature the knot quite a few more times. Note that it is thus *opposite* to Rolfsen's [Ro, Appendix] mirroring.

We also remark the following straightforward consequence of Theorem 3.6.

Corollary 3.8 When the grid diagram $!D$ is obtained from D by switching all crossings, and a $-\pi/2$ rotation, then $\lambda(D)+\lambda(!D)=\mu(D)$.

3.2 Application to strong quasipositivity

Let $TB(K)$ be the *maximal Thurston-Bennequin invariant* of K, an invariant often considered in contact geometry [Fe, FT, LvN, Ng, Ma, Ru3, Ta]:

$$TB(K) := \max\{\, TB(D) \,:\, D \text{ is a diagram of } K\,\}\,.$$

We also specify a region which will play an important role throughout the rest of the paper.

Definition 3.9 We define the *framing diagram* $\Phi(K)$ of K as a subset of $\mathbb{R}^2$ by

$$\Phi(K) := \{\,(\mu,t) \,:\, A(K,t) \text{ has a strongly quasipositive band representation on } \mu \text{ strands}\,\}\,.$$

The following result of Rudolph [Ru3, Proposition 1] then follows directly from Theorem 3.6. (Note our different sign convention for t.)

Corollary 3.10 When K is not the unknot, then

$$\lambda(K) := \min\{\, t :\, A(K,t) \text{ is strongly quasipositive}\,\} = -TB(K)\,, \tag{3.13}$$

and more precisely,

$$A(K,t) \text{ is strongly quasipositive} \iff t \geq -TB(K)\,. \tag{3.14}$$

With the identification (1.1), we note already here the known bound (see [FT, Fe, Ta]) from the Kauffman polynomial, which will play a major role below:

$$\lambda(K) \geq -\min\deg_a F(K)+1\,. \tag{3.15}$$

For the unknot K, we have

$$-TB(\bigcirc)=1 \text{ but } \lambda(\bigcirc)=0\,. \tag{3.16}$$

The problem with (3.13) there is that $A(K,0)$ has the empty positive band presentation (on two strands), but we do not consider this band presentation corresponding to a grid diagram. For this reason, the unknot will repeatedly require special attention below. Despite the identification (3.13), $\lambda(K)$ will occur so often, that it is better to maintain the notation and avoid writing the minus sign most of the time, even when we exclude $K=\bigcirc$.

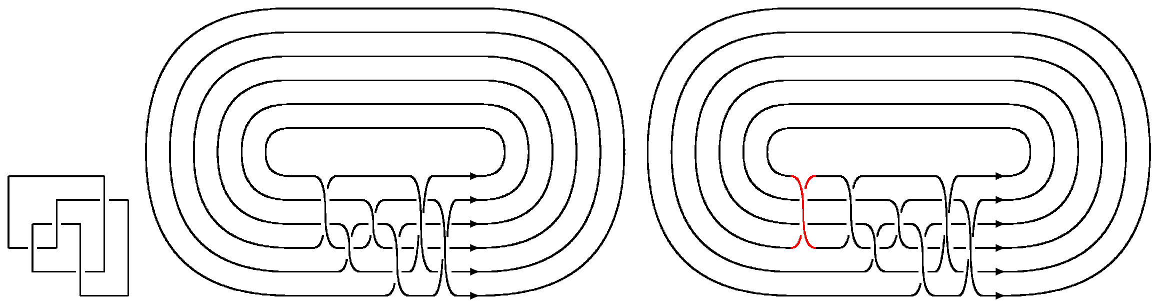
Remark 3.11 It is possible to derive similar properties for links K. Then a framing t is needed for each component, and the relationship in Corollary 3.10 becomes slightly more involved, as become the framing diagram of Definition 3.9 and its properties. We do not wish to deal extensively with links here. However, in situation where the surface structure is forgotten, the more self-contained extensions to links do emerge, as for Corollaries 4.4 and 4.5.

This is then a simple application of [Ru2]. We assume that $K \neq \bigcirc$. For $K=\bigcirc$, all the links in Definition 3.1 are (alternating) 2-bridge links, and such can be handled *ad hoc* for strong quasipositivity (see e.g. [Ba]).

Corollary 3.12 Let K be a non-trivial knot. Then
(a) $W_+(K,t)$ is strongly quasipositive if and only if $t \geq -TB(K)$, and
(b) $W_-(K,t)$ and $B(K,t)$ are never strongly quasipositive.

Since we will need this repeatedly later, let us already here notice that the Hopf plumbing $W_+(K,t) = A(K,t) * H$ can be realized by doubling a(ny) positive band in a band presentation of $A(K,t)$.

Example 3.13



4_1 $A(4_1, 3)$ $W_+(4_1, 3)$

A similar remark applies to $W_-(K,t)$ whenever a band presentation of $A(K,t)$ has a negative band. However, it is important to note that this is not the only way to generate positive band presentations of Whitehead doubles. (A different example for a trefoil Whitehead double is given in [Be, fig p. 121 bottom].)

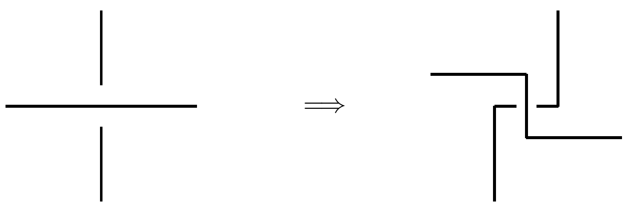
Then, we gave a simple application of the weight model, in estimating the Thurston-Bennequin invariant. A counterpart will emerge with Lemma 5.4 from the HOMFLY-PT polynomial.

Definition 3.14 Define $pbr(D)$, the *plane-bridge number* of D as the minimal number of Morse maxima (or minima, i.e., half of the minimal number of Morse extrema) over all smooth diffeomorphic images of D in S^2.

Obviously $br(K) \leq pbr(D)$ for the *bridge number* $br(K)$ of K (see e.g. [Mu]), and $br(D) \geq pbr(D)$ for every grid diagram D, where $br(D)$ was as defined in (3.6).

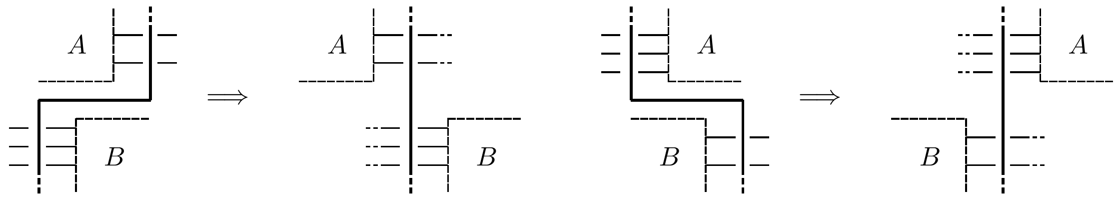
Lemma 3.15 For any diagram D of K, we have $\lambda(K) \leq 2c_-(D) + pbr(D)$.

We do not repeat the proof here, but we recall that it involved the *crossing conversion*



(3.17)

and *horizontal adjustment* technique.



(3.18)

4 Braid indices

We discuss here some remarks in [JLS] on the relation regarding the braid indices in Definition 2.1. (Compare with [Nu, Section 3.3].) As noticed, Bennequin's inequality (2.6) implies that a strongly quasipositive surface is a Bennequin surface, thus for K strongly quasipositive, we have (2.10). We know that $b_b(K) > b(K)$ is possible [HS], but the examples K known are not strongly quasipositive. Rudolph conjectures that

$$b_{sqp}(K) = b(K) \tag{4.1}$$

when K is strongly quasipositive, and this is true, among other families, if K is a prime knot of up to 16 crossings (see [St2]). By the proof of the Jones-Kawamuro conjecture (Theorem 2.2), a Bennequin surface of a strongly quasipositive link K on $b(K)$ strands is always strongly quasipositive, so that

$$b_b(K) = b(K) \tag{4.2}$$

implies (4.1) for strongly quasipositive knots K. The problem (4.2) is extensively studied in [St2].

Since a band presentation of $A(K,t)$ always comes from a grid diagram of K, and with a confirmative notice about the unknot, we have:

Corollary 4.1

$$\min\{b_b(A(K,t)) \,:\, t \in \mathbb{Z}\} = a(K)\,. \tag{4.3}$$

Moreover, there are at least $a(K)+1$ consecutive integers t which realize the minimum.

Also, because choosing positive bands will give a band presentation of a strongly quasipositive annulus, we have with Corollary 3.10:

Corollary 4.2 $\min\{b_{sqp}(A(K,t)) \,:\, t \geq \lambda(K)\} = a(K)\,.$

Forgetting the surface structure then yields an inequality of (ordinary) braid indices:

Corollary 4.3

$$\min\{b(A(K,t)) \,:\, t \in \mathbb{Z}\} \leq \min\{b(A(K,t)) \,:\, t \geq \lambda(K)\} \leq a(K) \tag{4.4}$$

Moreover, there are at least $a(K)+1$ consecutive integers t which realize the inequality $b(A(K,t)) \leq a(K)$.

The braid index of a link $A(K,t)$ is obviously not less than the sum of the braid indices of constituent components. Thus from Corollary 4.3, we also immediately have an inequality, which was noticed by Cromwell [Cr].

Corollary 4.4 (Cromwell) For every knot K, we have $2b(K) \leq a(K)$.

We observed then the (slight) refinement of Ohyama's inequality [Oh].

Corollary 4.5 For every knot K, we have $b(K) \leq c(K)/2+1$, and if K is non-alternating, then $b(K) \leq c(K)/2$.

These useful implications are worth noting, but we will see below that it is much more important to work with (4.4) rather than its simplified variant of Corollary 4.4. However, these simplifications have the advantage of extending with far less caveats to links K (see Remark 3.11).

We are next going discuss what (say, strongly quasipositive) framings λ are possible for given grid size μ, and in particular whether λ_{min}, the framing for a minimal (size $a(K)$) diagram (see Theorem 3.5) is unique. Since μ bounds the braid index of $A(K,t)$, and all have the same χ, Birman-Menasco [BM] imply that for given λ, only finitely many μ are possible. We will later prove a more precise statement (Finite-Cone-Theorem 5.3).

Question 4.6 (a) Is $b(A(K,t)) \geq a(K)$ for any t?
(b) At least, is $b(A(K,t)) \geq a(K)$ for any strongly quasipositive $A(K,t)$?

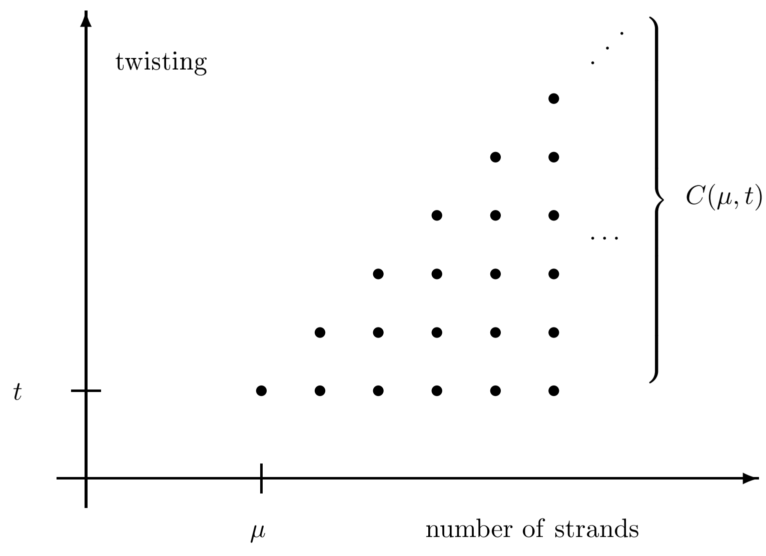
If (b) fails, then it would give an example $A(K,t)$ answering negatively Rudolph's question (4.1). This question will be further treated in Remark 5.24 and Conjecture 6.1.

To formalize this topic better, we introduced notation relating to the two grid stabilizations (3.9) and (3.10).

Definition 4.7 We define the *cone* $C(\mu,t) \subset \mathbb{Z}_+ \times \mathbb{Z}$ by

$$C(\mu,t) = \{\ (s,\lambda) :\ s \geq \mu,\ t \leq \lambda \leq t+s-\mu\ \}.$$

We say (μ,t) is the *tip* of the cone.



As argued, the framing diagram $\Phi(K)$ of K (see Definition 3.9) is a union of cones. We announced that we will prove later (Finite-Cone-Theorem 5.3) that cones in $\Phi(K)$ are finitely many. The following Jones-Kawamuro type of conjecture (compare with Theorem 2.2) is then suggestive.

Question 4.8 If K is non-trivial, is $\Phi(K)$ a single cone? (This cone would have to be then $C(a(K),\lambda_{min}(K))$ with $\lambda_{min}(K)=\lambda(K)$.)

Example 4.9 According to (3.7), we have

$$\Phi(\bigcirc) = C(2,0) \cup C(2,1)$$

being the union of two cones.

The special case for $\mu = a(K)$ in Question 4.8 (an analogue of the "weak" form of the Jones-Kawamuro conjecture) was already raised in [Ng] in the language of grid diagrams D and Thurston-Bennequin invariants $TB(D)$. It was answered in [DP, Corollary 3].

Theorem 4.10 (Dynnikov-Prasolov [DP]) The Thurston-Bennequin invariant of minimal grid diagrams of a given knot K is always equal to $TB(K)$.

We will return to this statement in §5.4 and §6.1. Note that the unknot creates no exception here, when using TB instead of λ and avoiding the discrepancy (3.16). Despite its importance, we do not use Theorem 4.10 substantially; it brings only minor simplifications, which can be mostly worked around. However, see (5.68), and its application in Example 5.42.

5 HOMFLY-PT polynomial

5.1 Some degree inequalities

We now turn our attention to the HOMFLY-PT polynomial P in (2.13). Our goal is to use the polynomial to prove that when t is sufficiently small, then $A(K,t)$ is not strongly quasipositive with a good lower bound on t. The $w(D)$ term in (3.5), as we have seen, makes bounds somewhat inelegant and inefficient. We use some notation from §2.4.

Lemma 5.1 For every knot K, there exists a strongly quasipositive framing $t = \lambda_{min}(K) \geq \lambda(K)$ of $A(K,t)$, so that

$$\min\deg_v P(A(K,t)) \geq 1\,, \quad \max\deg_v P(A(K,t)) \leq 2a(K) - 1\,. \tag{5.1}$$

Proof. When $K = \bigcirc$, then $t = 1$ suffices. Thus assume again below that K is non-trivial. When L is strongly quasipositive, then (2.8) and L being Bennequin-sharp mean that the right inequality in (2.7) becomes an equality. By using the right inequality in (2.17), we have

$$\min\deg_v P(L) \geq 1 - \chi(L) = 1 - \chi_4(L)\,. \tag{5.2}$$

In particular for $L = A(K,t)$, we have $\chi(L) \leq 0$, so

$$\min\deg_v P(A(K,t)) > 0\,. \tag{5.3}$$

We have from the skein relation (2.13)

$$P(A(K,t)) = v^2 P(A(K,t-1)) + vz\,. \tag{5.4}$$

Notice, by further remarks from §2.4, that for the 2-component link $A(K,t))$ the only monomials in $P(A(K,t))$ that occur are $z^p v^r$ with odd p,r. Also $\min\deg_z P(A(K,t)) = -1$, and by [LiM] it is known that

$$[P(A(K,t))]_{z^{-1}} = v^{2t}(v^{-1} - v)([P(K)]_{z^0})^2 \neq 0\,. \tag{5.5}$$

We now know that there is a (at least one) framing (we denoted) $t = \lambda_{min}$, so that $b(A(K,t)) \leq a(K)$.

Also by MFW inequality (2.18) we have

$$\mathrm{span}_v P(A(K,t)) \leq 2(a(K) - 1)$$

for $t = \lambda_{\min}$. Now, the diagram D_1 of $A(K, \lambda_{\min})$ obtained from a minimal grid diagram D of K by replacing vertical segments by positive bands has $w(D_1) = \mu(D) = a(K)$ and $s(D_1) = \mu(D) = a(K)$.

Thus by MFW inequalities (2.17), we have

$$\min\deg_v P(D_1) \geq 1\,, \quad \max\deg_v P(D_1) \leq 2a(K) - 1\,. \tag{5.6}$$

□

Lemma 5.2 If $K \neq \bigcirc$,

$$\lambda(K) > \max\{\lambda(D) - \mu(D) \,:\, D \text{ is a grid diagram of } K\}\,, \tag{5.7}$$

with non-strict inequality if $K = \bigcirc$.

Proof. By using the right inequality (5.1) and the recursion (5.4) reversely $a(K)$ times, we see

$$\min\deg_v P(A(K, \lambda_{min} - a(K))) \leq \max\deg_v P(A(K, \lambda_{min} - a(K))) \leq -1\,,$$

so from (5.3), we have that

$$A(K, \lambda_{min} - a(K)) \text{ is not strongly quasipositive,}$$

if $K \neq \bigcirc$. For $K = \bigcirc$, we can conclude that

$$A(K, \lambda_{min} - a(K) - 1) \text{ is not strongly quasipositive .}$$

In a similar way, for every grid diagram D of size $\mu(D)$, the annulus $A(K, \lambda(D))$ will appear in a diagram D_1 with $w(D_1) = s(D_1) = \mu(D)$, so

$$A(K, \lambda(D) - \mu(D)) \text{ is not strongly quasipositive} \tag{5.8}$$

when $K \neq \bigcirc$, and same for $A(K, \lambda(D) - \mu(D) - 1)$ when $K = \bigcirc$. □

Since this maximum is finite, we have:

Theorem 5.3 (Finite-Cone-Theorem) The framing diagram $\Phi(K)$ is a union of finitely many cones.

Proof. Note that a cone $C' = C(\mu', t')$ contains a cone $C = C(\mu, t)$ if and only if $(\mu, t) \in C'$. Thus if $C \subset \bigcup\limits_i C_i$, then $C \subset C_{i_0}$ for some C_{i_0}.

Call a cone $C \subset \Phi(K)$ *essential* if there is no cone $C' \subset \Phi(K)$ with $C \subsetneq C'$. Now consider the essential cones $C_i = C(\mu_i, t_i)$ in $\Phi(K)$ one by one. Order them as a (first potentially infinite) sequence $C_1, C_2, \dots$ by increasing t_i, i.e., so that $t_i > t_{i-1}$. Note that there cannot be two essential cones C_i, C_j with $t_i = t_j$, since otherwise $\mu_i < \mu_j$ would lead to $C_i \supset C_j$. Also there is a smallest t_1 because $t_i \geq \lambda(K)$ for all i. Define then

$$\nu_i = \max\{t - \mu \,:\, (\mu, t) \in C_i\}\,.$$

And now argue that $\nu_i > \nu_{i-1}$. Because of (5.7), there can be only finitely many increases of ν_i. (See Proposition 6.6 for a more precise statement and argument.) □

Another application of (5.7) gives an inequality we promised in stark symmetry to Lemma 3.15. (Unlike its counterpart, it thus does rely on the HOMFLY-PT polynomial in an essential way, though.)

Lemma 5.4 For any diagram D of K, we have $\lambda(K) > -2c_+(D) - pbr(D)$.

Proof. If $K = \bigcirc$, then $\lambda(K) = 0$, $pbr(D) > 0$ and $c_+(D) \geq 0$, so the inequality is trivial. Thus assume $K \neq \bigcirc$. We use the conversion (3.17) and the horizontal adjustment (3.18) of the proof of Lemma 3.15. We may then assume without loss of generality that D is a grid diagram and all ± 1 signed horizontal edges are intersected by a crossing. Then using (5.7), we have

$$\begin{aligned}
\lambda(K) &> \lambda(D) - \mu(D) \\
&= Z(D) - w(D) - \mu(D) \\
&\geq \frac{1}{2}\big(2\mu(D) - 2c(D) - 2br(D)\big) - \mu(D) - w(D) \\
&= -c(D) - br(D) - w(D) \\
&= -2c_+(D) - br(D)\,.
\end{aligned}$$

In the third line we used that each -1 edge has a crossing, and there are $2br(D)$ sign 0 edges. □

Remark 5.5 The number $l(K)$, introduced later, allows for improvements of (5.8), (5.7) and Lemma 5.4. However, the present versions maintain the advantage of involving only simple geometric data of the diagram itself, without protruding algebraic constructions derived from it. Since we will find a number of (other) applications of $l(K)$, we do not like to return here to resume this specific line of argument. The quantity $l(K)$ will serve as a lower estimate for the arc index, of which we put ahead a simplified variant.

Let $P = P(A(K,t))$ for some t. Keep in mind by §2.4 that $P|_{z^{\neq 1}}$ is the polynomial P with all terms of z-degree 1 removed. Because of (5.5), talking about its degrees makes sense.

Lemma 5.6 The integer

$$l'(K) := \frac{1}{2}\operatorname{span}_v P(A(K,t))|_{z^{\neq 1}} + 1 \tag{5.9}$$

does not depend on t and satisfies

$$a(K) \geq l'(K)\,. \tag{5.10}$$

Proof. By construction, $b(A(K, \lambda_{min})) \leq a(K)$, so by MFW inequality (2.18), we see that (5.10) is true for $t = \lambda_{min}$. And for other t, note that the relation (5.4) does not add any terms of z-degree different from 1. That $l'(K)$ does not depend on t follows for this same reason. □

But in fact, the z^1-term of P is also interesting – and significant – and its study relates to the "cooking" alluded to in the abstract of the paper.

5.2 Estimating $a(K)$: the pan

5.2.1 Definition and basic properties of the l-invariant

Like for the crossing number, there are only finitely many knots of given arc index.

The "classical" lower *Morton-Beltrami bound* for $a(K)$ comes from Kauffman's polynomial F [MB]:

$$a(K) \geq \mathrm{MB}(K) := \operatorname{span}_a F(K) + 2\,. \tag{5.11}$$

This can also be obtained from the bound (3.15) and Matsuda's inequality (5.67).

However, once such classical tool fails to give a sharp lower estimate, the method used so far, like in [J+], is to exhaustively enumerate all grids of smaller size, a feat which quickly becomes laborious and unreliable when the size increases. To change this situation here, we explain next how *not* to discard the z^1-term in (5.10), and use it to determine $a(K)$, and later $\lambda(K)$, from the P polynomial with considerable precision.

Write in the rest of this section for simplicity

$$K_t = A(K,t)$$

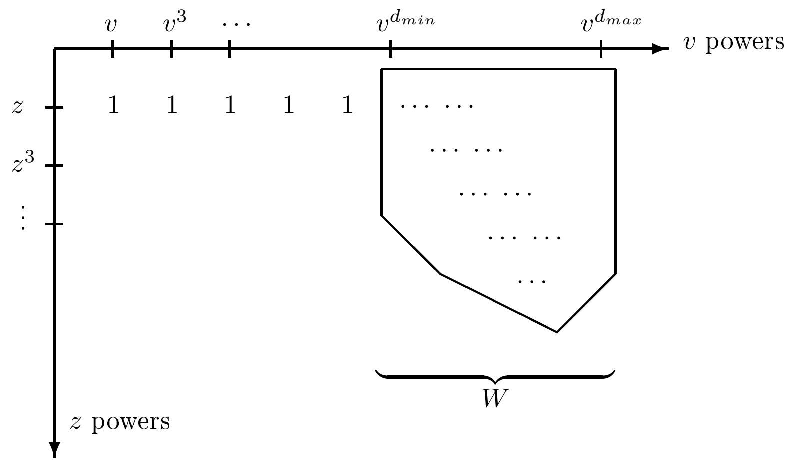
for (the boundary of) the t-framed annulus around K. We have then from (5.4),

$$P(K_t) = zv + v^2 P(K_{t-1})\,. \tag{5.12}$$

To visualize the polynomial $P(K_t)$, it will be helpful to plot its coefficients in the plane, with (odd) v degrees going from left to right and z degrees going top-down. Thus negative v-degree terms, left on the y-axis, occur, and will be considered. But negative z-degree terms, above the x-axis, occur only for z^{-1}, and we stipulate to hide them. We emphasize again that the z^{-1}-term in $P(K_t)$ is known (see (5.5))

$$[P(K_t)]_{z^{-1}} = (v^{-1} - v)\cdot v^{2t}([P(K)]_{z^0})^2\,. \tag{5.13}$$

By iterating (5.12), we can see that for sufficiently high t, the polynomial $P(K_t)$, displayed as we just explained, starts exhibiting the pan-like shape



(5.14)

Now remove all 1's in the panhandle of (5.14). To formalize this, we consider the leftmost and rightmost column $[P]_{v^d}$ in (5.14), for the smallest $d = d_{min} > 0$ which is not of the shape

$$[P]_{v^d} = z\,, \tag{5.15}$$

and

$$d_{max} = \max \deg_v P\,. \tag{5.16}$$

We can easily treat arbitrary t, and will do below. In that case, we can modify the condition (5.15) for $d_{max} < 0$ (keep in mind that for a 2-component link, d is always odd) to

$$[P]_{v^d} = -z \tag{5.17}$$

and $d_{min} < 0$ is set as $\min \deg_v P$. But, keeping the pan shape (5.14) in mind, assume here for simplicity $t \gg 0$.

Write then

$$l(K) = \frac{d_{max} - d_{min}}{2} + 1 \tag{5.18}$$

for the *(pan) width* of W in (5.14). (For the formalization of this procedure, see the expressions given at the end of §5.4. Compare also with [Nu, Theorem 3.3].) In result, we have a way to "normalize out" the t-dependence of the degrees of $P(K_t)$ in the z^1-term, giving an improved version of the lower bound $l'(K)$ in (5.10) for $a(K)$. Due to the attention incited by the unknot, let us remark here that

$$a(\bigcirc) = l(\bigcirc) = l'(\bigcirc) = 2\,. \tag{5.19}$$

Theorem 5.7 With (5.9) and (5.18), for every knot K, we have

$$l'(K) \leq l(K) \leq a(K)\,.$$

Proof. Obviously $l'(K) \leq l(K)$, so we prove the right inequality. Because of (5.19), we also assume $K \neq \bigcirc$.

When we set (5.16), it is possible that some $P(K_t)$ for small t has MFW bound $< l(K)$. This can happen if

$$[P]_{v^d} = z \text{ for } d = d_{max} \text{ and possibly } d = d_{max} - 2,\, d_{max} - 4, \text{ etc.} \tag{5.20}$$

In particular, we would need

$$d_{max} > \max \deg_v P|_{z^{\neq 1}} \tag{5.21}$$

for such terms to occur. These terms (5.20) can be cancelled by the inverse process of (5.12) when their v degree shifts down to 1 and then goes from 1 to -1.

We pause here for some cautionary illustrations. We do not know whether (5.20) can occur. But examples warn that it "almost" does. It can be seen from Table 1 that when $K = !10_{132}$, such a cancellation (when $t = 1$) occurs in degree $d_{max} - 2$. But it does not in degree d_{max}, which prevents a collapse in degree.

The polynomials from Table 2 are probably even more noteworthy (recall §2.4). By smoothing out a crossing in the Whitehead double clasp and taking the mirror image, one can see that when K are positive $(2,n)$-torus knots $T_{2,n}$, then terms $[P]_{v^d} = -z$ do occur in large amounts. The coefficients differ from (5.20) only by one wrong sign. (In this sense the "panhandle" they create is named "false"; it is an intrinsic feature of $K = T_{2,n}$ and does not come from large or small framing t in K_t.) These polynomials are peculiar enough to merit their own treatment later in Proposition 5.9.

In particular these "false" panhandles for $K = T_{2.n}$, also make a significant difference to $l'(K) = 4$ in (5.10), evidencing the price tag of ignoring the z^1-term all out. This is cemented by further knots like $K = 8_{20}, 9_{43}$, with $l'(K) < l(K)$.

Since we cannot exclude the situation (5.20), using

$$a(K) \geq \min_{t\in\mathbb{Z}} \text{MFW}(K_t) \tag{5.22}$$

(for (2.18)) will not be enough, at least in theory (see, though, Remark 5.8). However, notice that the arc index, as bound for $b(K_t)$, has a certain stability: there is a number $t = \lambda_{min}$ with

$$b(K_{t'}) \leq a(K) + |t' - \lambda_{min}| \tag{5.23}$$

for every t'. (We know by Theorem 4.10 that λ_{min} is unique for $K \neq \bigcirc$.) Using (5.23), we can replace (5.22) by

$$a(K) \geq \min_{t\in\mathbb{Z}} \max_{t'\in\mathbb{Z}} \text{MFW}(K_{t'}) - |t' - t|\,. \tag{5.24}$$

This will prevent the sporadic collapsing of the MFW bound from deteriorating the arc index bound. It can be seen, with a bit of technical argument based on (5.12), that the right of (5.24) is precisely what was defined as $l(K)$. This in particular shows

$$l(K) \geq \min_{t\in\mathbb{Z}} \text{MFW}(K_t)\,. \tag{5.25}$$

For instance, there can be at most two hypothetical values of t for which $\text{MFW}(K_t) < l(K)$, and for them choosing $|t' - t| = 1$ should suffice to see

$$\text{MFW}(K_{t'}) - |t' - t| \geq l(K)\,.$$

An instructive example of the argument, allowing for two such t to occur, is the following sequence. We show a transformation of the $[P(K_t)]_{z^1}$ terms with increasing t, where only the coefficients are written,

[25] [14] [37] [26] [15] [48] [79] [38] - [69]											
55	132	-1	17								
		7	13				9	-21	16	-4	
		5	15			-15	109	-186	86	31	-25
		1	15	-2	0	-80	452	-724	285	169	-100
		5	15			-148	870	-1493	659	272	-160
		5	15			-128	895	-1771	932	202	-130
		5	15			-56	520	-1256	772	76	-56
		5	15			-12	170	-536	376	14	-12
		5	15			-1	29	-134	106	1	-1
		7	11				2	-18	16		
		9	11					-1	1		
[25] [14] [37] [26] - [15] - [48] - [79] - [38] - [69]^2											
56	1	0	18								
		-2	4				-8	21	-16	4	
		-8	6	1	1	16	-108	186	-86	-31	25
		-8	6	2	0	80	-452	724	-285	-169	100
		-4	6			148	-870	1493	-659	-272	160
		-4	6			128	-895	1771	-932	-202	130
		-4	6			56	-520	1256	-772	-76	56
		-4	6			12	-170	536	-376	-14	12
		-4	6			1	-29	134	-106	-1	1
		-2	2				-2	18	-16		
		0	2					1	-1		

Table 1: Polynomial of the annulus link $A(10_{132}, 0)$ and the Whitehead double $W_-(10_{132}, -4)$ of 10_{132} and negative clasp, framing $t = -4$, together with the band presentation used, as obtained from (3.12) (where ±[ij] stands for $\sigma_{i,j}^{\pm 1}$ in (2.2), and the notation for polynomials follows §2.4).
The mirroring of 10_{132} can be easily confirmed from the z^{-1}-term of $P(A(10_{132}, 0))$ and (5.5) to be the one specified in Example 3.7.
For $A(10_{132}, 0)$ we see the disappearance of the (short) "false" panhandle. It comprises two monomials in z-degree 1. We call the panhandle "false" because in the same v-degree 1, a term $-2z^3v$ with z^3 occurs, so that this "panhandle" is not removed when reducing the framing t. Note that $A(10_{132}, 0)$ is not strongly quasipositive despite $\min \deg_v P > 0$.

and vertical bar stands for the separation between v-degrees -1 and 1 (making clear the degrees of all other coefficients; even degrees are obviously omitted). The pan edge coefficients (see (5.64)) are boxed at some places (similarly to (5.57); see also (5.66)).

$$\boxed{-1\ -1\ 2\ 0\ 0}\ -1\,\Big|\ \to\ \boxed{-1\ -1\ 2\ 0\ 0}\,\Big|\ \to\ -1\ -1\ 2\ 0\,\Big|1\ \to\ \dots \qquad (5.26)$$
$$\dots\ \to\ -1\,\Big|\,0\ 3\ 1\ 1\ \to\ \Big|\,\boxed{0\ 0\ 3\ 1\ 1}\ \to\ \Big|\,1\,\boxed{0\ 0\ 3\ 1\ 1}$$

In that case $l(K) = 5$, while for two t, $\mathrm{MFW}(K_t) = 3$ is possible. But for either t and one of the two t' with $|t' - t| = 1$, we have $\mathrm{MFW}(K_{t'}) = 6 = l(K) + 1$.

This argument based on (5.24) justifies that using (5.16) is appropriate to achieve $l(K)$ in (5.18) to estimate $a(K)$ as claimed. □

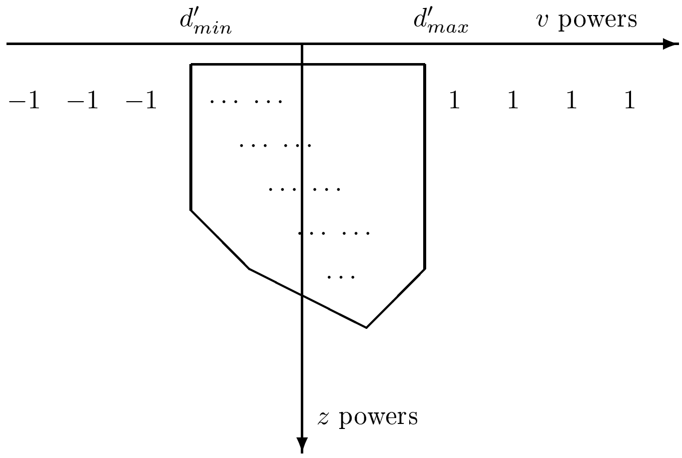
The equation (5.71) gives the formalization of the definition of l, which is postponed mainly due to its (here unnecessary) technicality. However, notice the following very self-contained special case (with the notation of (2.18)):

$$\text{when } t \text{ is chosen so that } \min\deg_v P(K_t) < 0 < \max\deg_v P(K_t), \text{ then } l(K) = \mathrm{MFW}(K_t). \qquad (5.27)$$

Remark 5.8 There is a way to modify the calculation of $l(K)$ to determine the right hand-side of (5.22) in practice. Remove all highest v-degree terms (5.15) for $d > 0$ and $[P]_{v^d} = 0$ for $d < 0$, until you reach a degree d'_{max} (with coefficient $[P]_{v^d}$) not of that form. Similarly, remove all lowest v-degree terms (5.17) for $d < 0$ and $[P]_{v^d} = 0$ for $d > 0$, finding d'_{min}. Then (5.25) can be extended to

$$l(K) \geq \min_{t\in\mathbb{Z}} \mathrm{MFW}(K_t) \geq \frac{d'_{max} - d'_{min}}{2} + 1\,. \qquad (5.28)$$

Note that on the right there is still no equality, because when t is fixed, the just described cancellation of terms in $P(K_t)$ can only occur on one side (either for low, or for high powers of v, but not for both). That is, we may in theory have a situation like

$P(K_t) =$ 

, (5.29)

which is also the type that occurs in (5.26).

Still, in the present form the estimate (5.28) is good enough to allow us to confirm that in fact

$$l(K) = \min_{t\in\mathbb{Z}} \mathrm{MFW}(K_t) \qquad (5.30)$$

(i.e., (5.29) does not arise, and (5.25) is exact) for all prime knots K up to 10 crossings. We do not know whether this equality holds in general.

Before moving over to some general properties of the l-invariant, it is worth paying the due tribute to the particular polynomials in Table 2, which are striking enough to merit an explicit statement.

Proposition 5.9 For the $(2,n)$-torus knot $T_{2,n}$, the polynomial $P((T_{2,n})_{-n})$ is of the shape

$$P((T_{2,n})_{-n}) = \tilde{P}_n - zv^5 \frac{v^{2n-4}-1}{v^2-1}\,, \tag{5.31}$$

where

$$-3 = \min\deg_v \tilde{P}_n \le \max\deg_v \tilde{P}_n = 3\,. \tag{5.32}$$

In particular, $l(T_{2,n}) = n+2$, while $l'(T_{2,n}) = 4$ (when $n>1$).

Proof. The skein algebra of rooms with 4 inputs and 4 outputs has $4! = 24$ linear generators. Thus the polynomials $P((T_{2,n})_{-n})$ satisfy some linear recursion of length 24. What we claim is (mostly) that

$$P_n = v^{3/2}P((T_{2,n})_{-n})(v^{1/2},z) + zv^4\frac{v^{n-2}-1}{v-1}$$

satisfies

$$P_n^* = \frac{\partial^4}{\partial v^4}P_n = 0 \tag{5.33}$$

for all n. These polynomials P_n^* have a recursion of length $24\cdot 5 = 120$ (where '5' is one plus the degree of derivative). Thus it is enough to prove that $P_n^* = 0$ for 120 consecutive odd n. By using mirror images, we can reduce this number to the 60 values of $n = 1,3,\dots,119$. (We observed from the below computation that the situation for even n, where $T_{2,n}$ is a link, is slightly more complicated.)

Thus if we prove (5.31) for $1 \le n \le 119$ odd, we would be done. For this, we can resort to an explicit computation using the skein algebra. (A naive skein relation (2.13) calculation from the resulting diagrams would take too long.)

It is a module of rank 24 over $\mathbb{Z}[v^{\pm1},z^{\pm1}]$, but is generated as an algebra by the 4 elements in [DM, p. 2964, l.12]. (Thus only multiplication with these 4 elements needs to be explained.)

This implementation (in C++) completed the test in 5 minutes. (The complexity is about quadratic in n, taking into account the growing skein module coefficients. Within 1 day, we were able to calculate and test the polynomial until $n \approx 530$.)

Pay attention that (5.33) will show only

$$-3 \le \min\deg_v \tilde{P}_n \le \max\deg_v \tilde{P}_n \le 3$$

in (5.32), but equalities easily follow from looking at $[P((T_{2,n})_{-n})]_{z^{-1}}$ and using (5.5). □

Notice that [DM, Theorem 2.7] exhibits (by a hand argument) only the term in maximal v-degree $2n-1$ in (5.31). (The sign convention for the framing there is different, see below Definition 3.1, and either is the normalization of P.) It is hard to identify the (even "false") panhandle from its "tip" only. But in turn, there seem deeper reasons (see below) that a computer obscures. We will return to discussing Diao-Morton's theorem in the proof of Proposition 5.17.

Example 5.10 One should be cautioned that such extended panhandles do not only arise for $T_{2,n}$. In forthcoming work of the second author with Mironov-Sati-Singh, we understood this panhandle property for a general torus knot $T_{p,n}$. In particular, we know that $T_{p,n}$ is l-sharp (see below), which gives a quantum-algebra proof of Etnyre-Honda's result $a(T_{p,n}) = p+n$. We also know that $\mathrm{MB}(T_{p,n}) = 2p$ (independently of n), when p is odd. However, the case of torus *links* remains to be studied.

30	1	0	14											
	-2	4							9	-33	41	-16		
	-12	4		1	1	1	1	1	25	-164	221	-80		
	-2	4							22	-342	468	-148		
	-2	4							8	-376	496	-128		
	-2	4							1	-231	286	-56		
	0	4								-79	91	-12		
	0	4								-14	15	-1		
	0	2								-1	1			
38	1	0	18											
	-2	4									16	-56	66	-25
	-16	4		1	1	1	1	1	1	1	81	-420	541	-200
	-2	4									148	-1316	1778	-610
	-2	4									128	-2248	3040	-920
	-2	4									56	-2298	3013	-771
	-2	4									12	-1457	1821	-376
	-2	4									1	-575	680	-106
	0	4										-137	153	-16
	0	4										-18	19	-1
	0	2										-1	1	

Table 2: Polynomials of the Whitehead doubles $W_+(7_1, 7)$ and $W_+(9_1, 9)$ of the negatively mirrored 7_1 and 9_1. The framing t can be read off, because of (5.55), from the sum of the coefficients in the second row.
It should be emphasized that what appears as a panhandle is *not* what is illustrated in (5.14). It is at the "wrong" end and will remain part of the pan when t is large.
Had the coefficients in these "false" panhandles been signed in the *opposite* way, i.e., to be -1, the polynomials of $A(!7_1, t)$ and $A(!9_1, t)$ would have instantiated the possibility (5.20). (Being signed $+1$, these coefficients will become 2 for large t.)

5.2.2 Estimates and l-sharpness

Since $P(A(K,t))$ are interconvertible for all t, one can determine $l(K)$ from $P(A(K,t))$ for any t, and then hope to determine $a(K)$ if $a(K) = l(K)$.

Definition 5.11 We say that K is *l-sharp* if $a(K) = l(K)$, and l-unsharp otherwise.

Example 5.12 Among the Rolfsen [Ro, Appendix] knots, $K = 10_{132}$ is the only one which is not l-sharp. Then $l(K) = 8$ (as shown in Table 1) but [J+] (see Example 3.7) and KnotInfo [LvM] report $a(K) = 9$.

There are four l-unsharp 11 crossing knots K (up to mirror images), i.e., such with

$$a(K) > l(K)\,, \tag{5.34}$$

namely 11_{379}, 11_{424}, 11_{455}, 11_{459} (for which $l(K) = 9$ and $a(K) = 10$), and 21 further examples of 12 crossings.

In case of 10_{132} (and a series of other examples), there is a linking number argument that can help out determining the arc index. We formulate it as a lemma. (It can also be easily modified to other knots, but for simplicity we just present the prototype and leave it to the reader to adapt it.)

Lemma 5.13 We have $a(10_{132}) = 9$.

Proof. Assume $a(10_{132}) \le 8$. From the polynomial of the annulus link $A(10_{132}, 0)$ in Table 1, and (5.4), we can see that $\mathrm{MFW}(A(10_{132}, t)) \le 8$ occurs for $t = -8, \dots, 0$, and then $\mathrm{MFW}(A(10_{132}, t)) = 8$. Because of the bottom statement of Corollary 4.3, it is enough to prove that $b(A(10_{132}, 0)) \ne 8$. We claim that the polynomial of $A(10_{132}, 0)$ in Table 1 is sufficient to see that $b(A(10_{132}, 0)) > 8$, as follows.

Assume $b(A(10_{132}, 0)) = 8$, and β is an 8-braid whose closure is $A(10_{132}, 0)$. Now, the exponent sum $w(\beta)$ is made up of the exponent sums $w(\beta_i)$ of the two subbraids β_i of β, which give the individual components $\hat\beta_1 = C_1$ and $\hat\beta_2 = C_2$ of $A(10_{132}, 0)$, and the linking number $lk(C_1, C_2) = t = 0$ of these components. Since both C_1 and C_2 have the knot type of 10_{132}, and $b(10_{132}) = 4$, both components C_1 and C_2 of $A(10_{132}, 0)$ must be closures of 4-string subbraids β_i of β. Then their individual exponent sums must be $w(\beta_i) = w_{min}(10_{132})$, which is determined to be 3 (see the tables [St4] and the remarks below (2.19)). Thus

$$w(\beta) = w(\beta_1) + w(\beta_2) + lk(C_1, C_2) = 3 + 3 + 0 = 6\,.$$

But the polynomial $P = P(A(10_{132}, 0))$ in Table 1 exhibits

$$\min\deg_v P = 1 \le 15 = \max\deg_v P\,,$$

and looking at the refined inequality (2.17), we see that a braid β with $n = 8$ strands must have writhe $w = w(\beta) = 8$. This is a contradiction. □

For all of the 26 l-unsharp knots of Example 5.12 we have $\mathrm{MB}(K) = l(K)$ in (5.11). But $\mathrm{MB}(K) < l(K)$ obviously occurs for some "F-sparse" knots like $K = 9_{42}$. (However, compare also with Example 5.32.) Likewise, $l'(K) < \mathrm{MB}(K)$ occurs in Table 2 (due to (5.37)), thus the z-term retains its credentials. See further Example 5.10.

Question 5.14 Is

$$\mathrm{MB}(K) \le l(K) \tag{5.35}$$

for all non-trivial knots K?

Example 5.15 In general the approximation

$$l(K) \leq a(K) \tag{5.36}$$

is rather good. There are 2049 arc index 11 prime knots up to 16 crossings. The inequality (5.11) is sharp for 1666 of them, while 1977 (including all those 1666) are l-sharp.

Remark 5.16 Usually, $F(K)$ is easier to obtain than $P(A(K,t))$. However, for small values of k, the truncation $P|_{z^{\leq k}}(A(K,t))$ may come out faster than $F(K)$. When $F(K)$ is too slow, this raises the issue of computing z-truncations thereof (since there are truncated versions of (5.11) as well). While the technology is implemented [St3] and ready to use, we choose not to delve into this here at all. As long as K is not excessively complicated, $F(K)$ is comparatively efficient to obtain, and thus, in practical terms, there seems little wrong to always try (5.11) first as a lower bound for $a(K)$. For suggestive reasons, (5.11) will accompany us constantly (see e.g. end of §5.3), but we like to focus on the HOMFLY-PT polynomial, and thus will not make the comparison to (5.11) everywhere.

For an *alternating* knot K, Yokota [Yo2] proved that

$$\text{span}_a F(K) = c(K)\,, \tag{5.37}$$

and we know by [BP] (as can be seen from the proof of Corollary 4.5 in §4) that (5.11) is an equality, i.e.,

$$a(K) = c(K) + 2 \tag{5.38}$$

for each such knot K. Thus in particular a positive answer to Question 5.14 must imply that K is l-sharp. After some preliminary supportive computations, we found that this is indeed true, and it provides a considerable motivation for studying the phenomenon outside of its intrinsic definition.

Proposition 5.17 Every alternating knot K is l-sharp.

Proof. This is essentially proved by Diao and Morton [DM], so the present proof is an explanation how to extract, adapt and simplify what of their work is needed here. We stipulate *within this proof* that numeration of *theorems* refers to [DM], while propositions, tables and equations to the present paper. In the case of notation, the default will be ours here, unless we specify otherwise.

Two main ingredients are needed. The first is Rudolph's congruence [Ru5] (Theorem 2.2), which relates $P(K_t)$ modulo 2 to $F(K)$. For the purpose of applying (2.18), write $\text{MFW}_{\text{mod }2}(L)$ for the bound obtained from $\text{MFW}(L)$ when the polynomial $P(L)$ has its coefficients reduced modulo 2, so that

$$\text{MFW}_{\text{mod }2}(L) \leq \text{MFW}(L) \leq b(L)\,.$$

(This notation is not to be confused with (2.20).)

Thistlethwaite has extended his proof of (5.37) in [Th], and the second ingredient is a refinement of Thistlethwaite's work, in the case of alternating links, due to Cromwell [Cr3] (Theorem 2.3). It exhibits (5.37) through explicit *odd* coefficients

$$[F(K)]_{z^{k_1}a^{l_1}} = [F(K)]_{z^{k_2}a^{l_2}} = 1 \quad \text{with} \quad l_2 - l_1 = c(K). \tag{5.39}$$

(Alternatively, see [Th, Corollary 1.1(iv)].)

When D is a reduced alternating diagram of K, and D is not a $(2,n)$-torus link diagram (we may assume n odd), then $k_1, k_2 > 1$. Via Theorem 2.2, this immediately implies that $\text{MFW}(K_t) \geq c(K) + 2$ for all t, and because of (5.25) and (5.38), this shows our claim.

However, if D is a $(2,n)$-torus link diagram, then one of k_1, k_2 in (5.39) is 1, and there is exactly one framing t for which $\text{MFW}_{\text{mod }2}(K_t) < c(K)+2$. Note that we have encountered this case in fact: these are the "false" panhandles of Table 2 and Proposition 5.9. Diao and Morton then engage in a tricky calculation (Theorem 2.7) to show that the "tip" of these panhandles is a '1' signed the wrong (i.e., uncancellable) way, so that still $\mathrm{MFW}(K_t) = c(K) + 2$.

But one of the nice features of our approach is that this complexity (or its computerized alternative from Proposition 5.9) is not needed here, when instead of (5.25) we use (5.27). With (5.39), one can obviously choose a framing (with the notation of [DM]) f so that the right hand-side of the congruence in Theorem 2.2, call it

$$R_f(v,z) = v^{2f}\left(-1 + \frac{v^{-1}+v}{z}\right) F(K)(v^2, z^2)$$

(keep in mind our opposite convention of both framing f and a-powers in F), has

$$\min\deg_v (R_f)_{\bmod 2}(v,z) < 0 < \max\deg_v (R_f)_{\bmod 2}(v,z)\,. \tag{5.40}$$

The congruence then implies (with our notation, but f retaining its role) that

$$\min\deg_v P(K_f) < 0 < \max\deg_v P(K_f)\,, \tag{5.41}$$

which with (5.37), (5.39) and (5.27) shows $l(K) \geq c(K) + 2$, as needed. □

Remark 5.18 To establish the minimality of the band presentations where (2.18) fails modulo 2, Theorem 2.7 of [DM] can be replaced not only by a computer calculation (Proposition 5.9), but also by a simpler (manual) argument using Bennequin's inequality (see the proof of Proposition 6.9). Thus, if one is still allowed to invoke [Cr3, Ru5], this gives a second alternative path towards the main result in [DM], this time without proving that (2.18) is exact on all K_t. The work in [DM, §3] is of course understood here as well, e.g., Corollary 4.1. We formulate in Proposition 6.14 the extended version of Diao-Morton's result we obtain.

The argument based on Rudolph's congruence implies that the answer to Question 5.14 is affirmative if in the definition $\mathrm{MB}(K)$ in (5.11), coefficients of F are reduced modulo 2. Notice that, historically, this weaker form of the inequality (5.11) had been previously discovered by Nutt.

The proof of Proposition 5.17 also yields a more generalized version, which is a very practical way to test Question 5.14.

Corollary 5.19 Assume $F(K)$ has odd coefficients $[F(K)]_{a^{l_i}z^{m_i}}$, $i = 1, 2$, so that $l_1 = \min\deg_a F$ and $l_2 = \max\deg_a F$. Then (5.35) holds for K.

Proof. The assumption means that

$$\mathrm{span}_{\,a} F_{\bmod 2}(K) = \mathrm{span}_{\,a} F(K)\,. \tag{5.42}$$

If $\mathrm{span}_{\,a} F(K) \geq 1$, then

$$\mathrm{span}_{\,v} (R_f)_{\bmod 2}(v,z) = 2 + 2\,\mathrm{span}_{\,a} F \geq 4\,, \tag{5.43}$$

and so we can find an f with (5.40), and the rest of the argument is the same as for Proposition 5.17, with '$c(K) + 2$' replaced by '$\mathrm{MB}(K)$'.

Now assume

$$\mathrm{span}_{\,a} F(K) = 0\,. \tag{5.44}$$

We only use that $\mathrm{MB}(K) = 2$, and remark that since $[P(K)]_{z^0} \neq 0$, from (5.5) we have $\mathrm{span}_{\,v} P(A(K,t)) \geq 2$, as needed. □

Remark 5.20 When (5.44), it can be argued that $F(K) = 1$. The first two properties enumerated at the end of §2.4 imply that $F(K) = z^k$ for some (integer) k. The third one (2.27), together with the known property $V'(-1) = 0$ for any knot K, shows that $k = 0$. But this is not needed for the proof.

Computation 5.21 The inequality (5.35) holds for all prime knots K up to 16 crossings. Among knots up to 10 crossings, only 9_{46}, 10_{142}, and 10_{160} fail the premise of Corollary 5.19. This suggests that it filters potential counterexamples rather well. Testing (only) non-alternating knots up to 15 crossings, we find that the assumption fails on 7,328 out of 201,702. They can then be dealt with through approximating $l(K)$ by successive truncations of $P(K_t)$. (The work took 2h on an old 2013 laptop.) For 16 crossings, see Computation 5.44.

Another application of Corollary 5.19 is when K is a torus knot: the inequality (5.35) holds for all such knots (not always as an equality). But torus knots require a longer separate treatment. Example 5.10 is a very partial indication.

We further know that for prime knots K up to 18 crossings, $l(K) \geq 4$. This uses Rudolph's congruence and was a benchmark test for the (z-)truncated F method of [St3]. (It required 4h. To save space, we do not delve into this verification further. We also ascertained that $\mathrm{MB}(K) \geq 5$ in that range, but with Example 5.10, less improvement can be expected there.)

In general, $l(K)$ is not easy to calculate on infinite families of knots. Notice that, unlike (2.28) and a corresponding property of the right hand-side of (5.11), it is not even evidently (2 sub-)additive under connected sum.

Question 5.22 Is $l(K_1 \# K_2) = l(K_1) + l(K_2) - 2$?

This turns out to be the case in a few examples, like $10_{132}\#(!)3_1$ and $10_{132}\#(!)10_{132}$, but as long as it is not confirmed, the possibility exists to extract further information from l as a lower arc index bound, using the relationship (2.28). Still, the answer is affirmative if polynomial coefficients are taken modulo 2.

In view of this presumable behavior of $l(K)$, perhaps a reasonable expectation regarding (5.36) as an arc index bound is like this: is there a positive constant C with $C \cdot l(K) \geq a(K)$ for all K? Such constant does not exist for $\mathrm{MB}(K)$. Example 5.10 shows, in a way, that the worst-case performance of (5.11) is as bad as possible.

5.3 Applications of Cabling

Conjecture 2.3 underscores the importance of cabling in settling braid, and thus also arc index issues. This is a perhaps less pleasant, but still more universal means than Lemma 5.13, to treat some l-unsharp knots K.

Computation 5.23 For $K = 10_{132}$ the links L we consider with $\mathrm{MFW}(L) = 8 < a(10_{132}) = 9$ are

- $L = A(K,t) = K_t$ for $t = 0, \ldots, -8$,
- $L = W_+(K,t)$ for $t = 0, \ldots, -7$, and
- $L = W_-(K,t)$ for $t = -1, \ldots, -8$.

(Of course, for the rest values of t we can conclude $\mathrm{MFW}(L) \geq 9$ using the relation (5.12), or a similar relation for Whitehead double polynomials.)

All the links listed above have $b(L) = 9$. We easily observe $b(L) \leq 9$. One can obtain a 9-string band presentation from that for $A(10_{132}, 1)$ with positive bands, given in (3.12), by making some bands negative and doubling a positive band for W_+ and a negative one for W_-. (Table 1 gives some examples.) At the opposite end, we tested $b(L) \geq 9$ with parallelized truncated 2-cable (MFW) P, as discussed in §2.4. The procedure took on a 4-CPU 10-year-old 2013 laptop between 2 and 15 h depending on individual examples: an agreeable performance, when taking into account that the diagrams resulting from 2-cabling the modifications of (3.12) have more than 200 crossings. (They depict $\uparrow\uparrow\downarrow\downarrow$ oriented degree-4 satellites of 10_{132}.)

This comparative efficiency offers the opportunity for more extensive checks (for other K). However, this option was waived on, since it still is not readily amenable to larger quantities, and it leaves unclear what insight to expect. (We will use the above compiled examples for later reference, though.)

Remark 5.24 Using Computation 5.23 for $K = 10_{132}$, and the verification of (5.30) and $l(K) = a(K)$ (see Example 5.12) for all other prime knots K up to 10 crossings, we can conclude that the answer to (both parts of) Question 4.6 is affirmative for all these 249 knots.

When (5.34) occurs, i.e., K is not l-sharp, the following simplification of cabling may potentially be useful. Since $\kappa(A(K,t)) = 2$, one can cable an individual component of $A(K,t)$, obtaining a $\uparrow\uparrow\downarrow$ oriented parallel $A^*(K,t,t')$ of K, where t' is the framing of the doubled component. (Here thus t' can be a half-integer when the two copies of the doubled component get connected, i.e., $\kappa(A^*(K,t,t')) = 2$ when $2t' \in \mathbb{Z}$ but $t' \notin \mathbb{Z}$.) Cabling an individual component only roughly doubles (and does not quadruple) the crossings in the braid word β_D for $A(K,t) = \widehat{\beta_D}$.

Lemma 5.25 For every t with $b(A(K,t)) = a(K)$ and every $2t' \in \mathbb{Z}$, we have

$$b(A^*(K,t,t')) \leq \frac{3}{2}a(K)\,. \tag{5.45}$$

Proof. When $b(A(K,t)) = a(K)$, then one of the components of $A(K,t)$ in an $a(K)$-braid representative β is a subbraid on at most $a(K)/2$ strands. Thus doubling this component C, regardless of what framing t' is used, can be done by adding at most $a(K)/2$ braid strands. (The framing can be corrected by adding half-twists which do not add more strands.) This gives a braid representative of $A^*(K,t,t')$ of at most $3a(K)/2$ strands, resulting in (5.45).

Note that $A(K,t)$ is exchangeable up to simultaneous reversal of orientation of both components, which does not affect braid index arguments. Thus whether C is the component we 2-cable to obtain $A^*(K,t,t')$ from $A(K,t)$, or we cable the other component, is not relevant. (Note, though, that the framing t' of the cabled component may be different with respect to the blackboard framing of the diagram $\hat{\beta}$.) □

Algorithm 5.26 The following explains how one can try to use this lemma. Since the contrapositive of its statement is really used, some care is needed how to proceed, and we formulate it in several steps as an algorithm.

1. Use a band presentation β_D (as in (3.1)) for a grid diagram D of K of size μ. This gives a band presentation of $A(K,t)$ for some t.

2. Make some bands negative to ascertain that $P(A(K,t))$ has no panhandle. For example, when $K = 10_{132}$ and $\mu = 9$, then we know that there are nine values of $t \in \mathbb{Z}$ for which $\mathrm{MFW}(A(K,t)) = l(K) = 8$, namely $t = -8, \dots, 0$. The statement below (4.4) says that it is enough to treat one of these t. Thus we can consider $t = 0$ (which requires one negative band), and use the polynomial in Table 1. In general, one can remove the panhandle (i.e., adjust t by making bands negative) only by looking at $P(A(K,t))|_{z\leq 1}$.

3. Then double, with blackboard framing with respect to the diagram $\widehat{\beta_D}$, one of the components of the link $\widehat{\beta_D} = A(K,t)$. One obtains a link $A^*(K,t,t')$. There are in general two possibly (but not always) distinct integers t', depending on which component of $\widehat{\beta_D}$ one chooses to double. (It can be argued that these two t' will add up modulo 2 to the same parity as the "band width" sum $\sum_{k=1}^{\mu}(j_k - i_k - 1)$ in (2.5); which in turn has the same parity as μ; thus two distinct t' will in particular always occur when μ is odd.)

4. Try to prove that such a link $A^*(K,t,t')$ has braid index strictly greater than $\lfloor 3(\mu-1)/2 \rfloor$. This will prove $a(K)=\mu$.

Example 5.27 For instance, when we do this construction for $K=10_{132}$ with (3.12) (one band needs to be made negative here), this gives $A^*(10_{132},0,t')$ for $t'=3,4$. We found (see (2.20)), though, that

$$\mathrm{MFW}_{10}(A^*(10_{132},0,t'))=12$$

for both t'. Thus unfortunately, for $K=10_{132}$, the observation (5.45) does not seem useful to show $a(10_{132})=9$, at least as far as (2.20) is applied (within reasonable computability).

However, there is a number of successful cases. For example, when we carry out this process for $K=14_{27072}$, with the size-12 grid

```
13  24  58  7C  3B  1A  6C  59  8B  7A  49  26
```

(where A,B,C stand for 10,11,12; see Definition 3.4), we find $l(14_{27072})=11$, but making 3 bands negative, we obtain

$$\mathrm{MFW}_2(A^*(14_{27072},2,1))=17$$

(here $t'=1$ is the same for both choices of doubled component), which rules out $a(14_{27072})=11$.

Other examples, again with $\mu=12$ (and a single t'), are

```
16 466746: 13  46  25  7A  8B  9C  3A  4B  16  7C  28  59
15 123702: 13  24  57  9C  6A  38  17  5B  49  8C  2A  6B
```

and

```
14 19935:  13  25  48  7B  3A  16  59  8B  7A  49  26
16 459158: 14  25  38  6A  7B  49  18  5A  29  6B  37
```

for $\mu=11$ (using 5 negative bands, with two different t', both having $\mathrm{MFW}_2(A^*(K,t,t'))=16$).

These examples do require some search, but keep in mind that even for truncated polynomials, the increase in crossing number has severe (complexity) consequences. (Here we tried only truncation degree $d=2$, which does not cost much time and allows for testing a larger number of examples.) Thus Lemma 5.25 provides a viable option to try out.

Remark 5.28 We add the following practical hints about the determination of the arc index.

1) For more complicated knots K, it is better to approximate $l(K)$ from below by using z-truncations of the HOMFLY-PT polynomial, as explained in §2.4. This was used to assist the first and third authors' ongoing effort to tabulate the arc indices of the (non-alternating prime) 14 crossing knots. But it also emphasizes that it is useful to have a good upper estimate of $a(K)$ in advance. Once coincidence with the lower bound is reached, one can then save calculation of further truncations (and the full polynomial).

   We clarify that how an upper estimate of $a(K)$ was obtained relates to the (knot-spoke) method of [JP], finding certain proper non-alternating arcs in diagrams of K. It is not necessary (and takes extra effort) to obtain a minimal grid diagram explicitly.

2) As noticed while proving Lemma 5.13, the statement below (4.4) provides another significant short-cut to help determining $a(K)$ when $l(K)$ fails. For instance, to see (in an alternative way to Lemma 5.13) that $a(10_{132})\neq 8$, it suffices to calculate a (truncated) 2-cable polynomial of $A(10_{132},t)$ for only (any) one of the nine values of t that occur in the enumeration of Computation 5.23.

3) Observe that the linking number argument of Lemma 5.13 can be adapted to $A^*(K,t,t')$ as well. One has to consider instead of $lk(C_1,C_2)=t$ the total linking number of the components of $A^*(K,t,t')$, which is $2t+t'$ for $t'\in\mathbb{Z}$ (and $\kappa(A^*(K,t,t'))=3$) and $2t$ otherwise (when $\kappa(A^*(K,t,t'))=2$). We will give relevant examples at a separate place, where we discuss the arc indices of the 14 crossing knots.

4) Notice also Question 5.22 and the remarks below it.

To give a lookout at where we stand thus far, regarding the said at the beginning of §5.2, we have now gained a toolkit to rule out certain values of the arc index. We related it to a braid index (see Conjecture 6.1 below, although Part 2 of Remark 5.28 explains that we need a weaker statement), and then in turn to the HOMFLY-PT polynomial (compare Conjecture 2.3). These connections work out at least in a practical sense, which gives an approach to determine $a(K)$ for most K.

We finish the subsection on cabling with some remarks on the relation to arc indices of cables of K, and a prospective (new) use of the Kauffman polynomial.

Proposition 5.29

$$a(K_{t_0})=2a(K) \text{ when } w(D)=-t_0 \text{ is a writhe of a minimal grid diagram } D \text{ of } K. \tag{5.46}$$

Moreover, each such $w(D)$ satisfies

$$\max\deg_a F(K)+1+br(K)-a(K)\le w(D)\le \min\deg_a F(K)-1-br(K)+a(K)\,. \tag{5.47}$$

Also

$$\min\{\,a(K_t)\,:\,w(D)=-t \text{ satisfies (5.47)}\,\}\;=\;\min\{\,a(K_t)\,:\,t\in\mathbb{Z}\,\}\;=\;2a(K)\,. \tag{5.48}$$

Proof. For '$\ge$' in the first statement, notice that the arc index of a link is not less than the sum of arc indices of its components. To see equality, take a minimal size $a(K)$ grid diagram D of K and build the (disconnected) blackboard-framed 2-parallel of D with reverse orientation of both components. This gives a grid diagram of size $2a(K)$ of K_{t_0} for $t_0=-w(D)$. With the same reasoning, we have (5.48).

An issue with using (5.46) as an arc index obstruction is that one does not really know *a priori* well what t_0 would have to be. One way to restrict t_0 is from [JLS, §3]. A generally better alternative arises using a known value or estimates of $\lambda(K)$. The form (5.47) we offer uses Corollary 3.8 with $\mu(D)=a(K)$. Note further that $Z(D)\ge br(K)$, since rotating D by $-\pi/4$ would turn NW-corners into local maxima (and SE into local minima) of a Morse presentation of K. This obviously holds for NE (or SW) corners as well (when rotating by $\pi/4$), and shows

$$br(K)\le Z(D)\le a(K)-br(K)\,. \tag{5.49}$$

Then we have from (3.5), and (3.15), when $K\ne\bigcirc$, that

$$w(D)-Z(D)=-\lambda(D)\le-\lambda(K)\le\min\deg_a F(K)-1\,,$$

which yields

$$w(D)\le\min\deg_a F(K)-1+Z(D)\,. \tag{5.50}$$

Applying the argument on the mirror image $!D$ gives

$$w(D)\ge\max\deg_a F(K)+1-Z(!D)=\max\deg_a F(K)+1-a(K)+Z(D)\,. \tag{5.51}$$

Use now (5.49) in (5.50) and (5.51), which shows (5.47). (When $K=\bigcirc$, the claim is trivially checked.) □

Further notice that altering individual component orientation of a link does not change the arc index, and thus, for an unrestricted $t\in\mathbb{Z}$, we may regard here K_t as a disconnected 2-cable of K. This would also lend a meaning to K_t for a half-integer $t \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, as a connected 2-cable. This situation was considered by the first author and Takioka [LT], where they write $q = 2t$. Still, one must be careful with the sign switch of t that occurs. To avoid confusion, let us write $\hat{K}_t$ for the 2-cable of K with framing $t \in \frac{1}{2}\mathbb{Z}$, so that when $t \in \mathbb{Z}$, then $\hat{K}_t$ arises by reversing one component in K_{-t}.

From here we see that we can also "2-cable" (5.11).

Corollary 5.30

$$2 + \min \{ \operatorname{span}_a F(K_t) \,:\, t \in \mathbb{Z} \} \,\leq\, 2a(K)\,. \tag{5.52}$$

Proof. For $\operatorname{span}_a F$ as well, it is immaterial how individual link components are oriented, and thus $\operatorname{span}_a F(K_t) = \operatorname{span}_a F(\hat{K}_{-t})$. This is the reason why when minimizing over $t \in \mathbb{Z}$, one can replace K_t by $\hat{K}_t$. □

It is not necessary to explicitly calculate $F(\hat{K}_t)$ for more than two values $t \in \frac{1}{2}\mathbb{Z}$, since there are recurrence relations (analogous to (5.12)), which determine all other $F(\hat{K}_t)$ therefrom. Thus in practice, a constraint like (5.47) is not very helpful, and it seems a bit easier to use $t \in \mathbb{Z}$ in (5.48).

Example 5.31 The first author and Takioka have employed this idea to determine $\operatorname{span}_a F(\hat{K}_t)$ for prime knots K of up to 8 crossings (and any $t \in \frac{1}{2}\mathbb{Z}$), and show that (5.11) can be used to find (*inter alia*) $a(\hat{K}_t)$ (and thus also $a(K_t)$ when $t \in \mathbb{Z}$) in all these cases. They did not consider $a(K)$, but their calculations [LT, Appendix A] establish that the practical variant of (5.52),

$$a(K) \geq 1 + \left\lceil \frac{1}{2} \min \{ \operatorname{span}_a F(K_t) \,:\, t \in \mathbb{Z} \} \right\rceil\,, \tag{5.53}$$

is sharp in their range. This was of course of little interest there, since $a(K)$ had long been determined previously. But it does motivate now a closer look at (5.53).

Example 5.32 Since (5.11) is not sharp for $K = 8_{19}$, there is some improvement from (5.53) over (5.11). In comparison to Theorem 5.7, the obvious instance to try out is again $K = 10_{132}$. It can be checked with some technicalities (of the same style as those handled by Lee and Takioka) that (5.53) is sharp for $K = 10_{132}$. (Still (5.52) is off by 1. Thus (5.11) does not yield enough information to determine $a(\hat{K}_t)$ for $K = 10_{132}$, at least when $t \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and the sublink argument at the beginning of the proof of Proposition 5.29 fails.)

This suggests the possibility that (5.53) is in fact quite powerful as an arc index bound. In how far (5.53) is useful in general remains to be seen. Certainly, when K has more crossings, the calculation of $F(\hat{K}_t)$ is very strenuous. But the truncation technique (Remark 5.16) could again come into effect.

Truncations could also become even more useful for higher cables. For instance, we can modify (5.53) to

$$a(K) \geq \left\lceil \frac{1}{3} \Big(2 + \min \{ \operatorname{span}_a F(A^*(K, -w(D), w(D))) \,:\, w(D) \text{ satisfies (5.47)} \} \Big) \right\rceil\,, \tag{5.54}$$

and here (5.47) becomes rather relevant again, since the recursions between $F(A^*(K, t, -t))$ (exist but) are much more cumbersome. Pay attention that (5.47) also involves $a(K)$, but this poses no problem in using (5.54) as an obstruction, in trying to falsify it when a particular value of $a(K)$ is fixed.

This approach does merit further study, but it definitely has to find its place in a separate paper, where we try it out on some 14 crossing knots.

5.4 Estimating $\lambda(K)$: a cooking recipe

Note the special form of the Conway polynomial (2.23) in our examples,

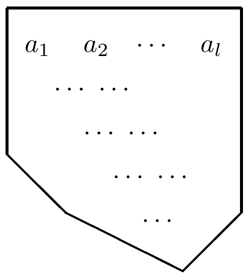
$$\nabla(K_t) = P(K_t)(1, z) = tz\,, \tag{5.55}$$

Returning to (5.14), we use the substitution (5.55) to extract further information from the pan.

Let $a_1, \dots, a_l$, for $l = l(K)$, be the z-degree 1 coefficients in W in (5.14):

$$[W]_{z^1} = \sum_{i=1}^{l} a_i v^{d_{min}+2i-2} . \tag{5.56}$$

Obviously a_i form the edge of the pan (drawn below without its handle) – whose general use is to break your eggs when frying them.



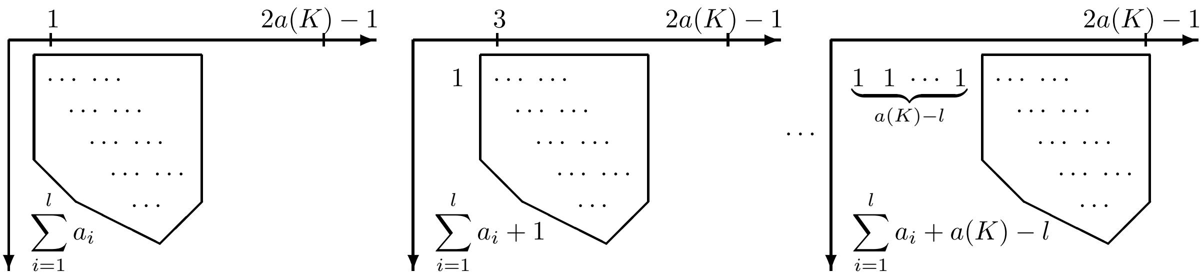
(5.57)

Note, though, that the possibility $a_1 = 0$ (or $a_l = 0$) does exist (although we did not investigate whether or how often it materializes). Furthermore, $a_0 = 1$ can occur also for $d_{min} > 0$ if $[P]_{v^{d_{min}}}$ has terms in z-degree $\neq 1$. Here is the way we put the pan edge to our own use.

Proposition 5.33

$$\sum_{i=1}^{l} a_i \leq \lambda(K) \leq \sum_{i=1}^{l} a_i + (a(K) - l(K)) . \tag{5.58}$$

Proof. Now remember that $\min\deg_v P(K_t) > 0$ (property (5.3)) for K_t strongly quasipositive (i.e., $t \geq \lambda(K)$), as well as that there is a $t \geq \lambda(K)$, namely $t = \lambda_{min}$, so that $\max\deg_v P(K_t) \leq 2a(K) - 1$ (property (5.1)). Thus, for the polynomial $P(K_{\lambda(K)})$ we have $a(K) - l(K) + 1$ possibilities



(5.59)

distinguished by the panhandle length $0, \dots, a(K) - l(K)$.

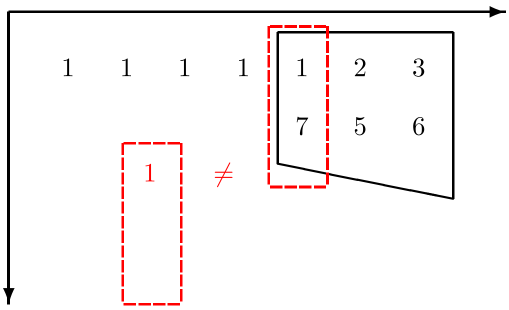
The pan edge coefficients a_i are not changed for different panhandle length, and by looking at (5.55), we see (5.58). □

Thus, rather precise, information about the Thurston-Bennequin invariant manifests itself in the coefficients of the polynomial, not in its degrees[1]. It provides an additional bonus of computing $P(K_t)$ (for some t), beyond determining $l(K)$. Namely, if $l(K) = a(K)$, then one obtains $\lambda(K)$ practically for free. This "frying eggs in the pan" procedure can be useful, for instance, in comparison to Theorem 4.10, when $a(K)$ is found without constructing a minimal grid diagram explicitly (see Part 1 of Remark 5.28), or as additional information in obstructing to the existence of certain grid diagrams of a given knot. Remark 5.41 gives a hint how to proceed when $l(K) < a(K)$.

To illustrate the use of (5.58), consider the following examples.

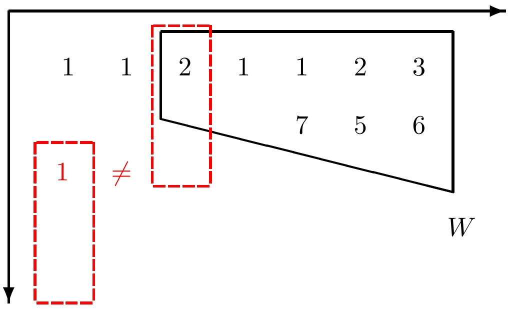
[1]Of course, if one is allowed to use $[P(K)]_{z^0}$, then t can be retrieved from $[P(K_t)]_{z^{-1}}$ using (5.13) as well.

Example 5.34 The polynomial[2]



has panhandle length 4 and pan-width $l(K) = 3$. If $a(K) = 5$, then (5.58) has on the right $(5-3)+(1+2+3) = 8$, so (5.58) reads $6 \le \lambda(K) \le 8$.

Example 5.35



has panhandle length 2 and pan-width $l(K) = 5$. If $a(K) = 6$, then (5.58) has on the right $(6-5)+(2+1+1+2+3) = 10$, so (5.58) reads $9 \le \lambda(K) \le 10$.

We have then the following "Matsuda-Dynnikov-Prasolov" (see Remark 5.38) type of relationship.

Proposition 5.36 With the notation of §2.2 for mirror image,

$$l(K) \le \lambda(K) + \lambda(!K) \le 2a(K) - l(K)\,. \tag{5.60}$$

Proof. We prove the right inequality. The argument can easily be modified to show the left one. We also assume, after inspection, that K is non-trivial. We have $(!K)_{-t} = !(K_t)$. Note that (2.15) (with $\kappa = 2$ as for $K_t = A(K,t)$) gives

$$P(!K_t)(v,z) = -P(K_t)(v^{-1},z)\,. \tag{5.61}$$

Now by mirroring property (5.1) using (5.61), we see that there is a $t = \lambda_{min}(K) \ge \lambda(K)$ with

$$\max\deg_v P((!K)_{-t}) \le -1\,, \quad \min\deg_v P((!K)_{-t}) \ge 1 - 2a(K)\,.$$

By how $l(K)$ was defined, and again using the mirroring (5.61), there is an odd

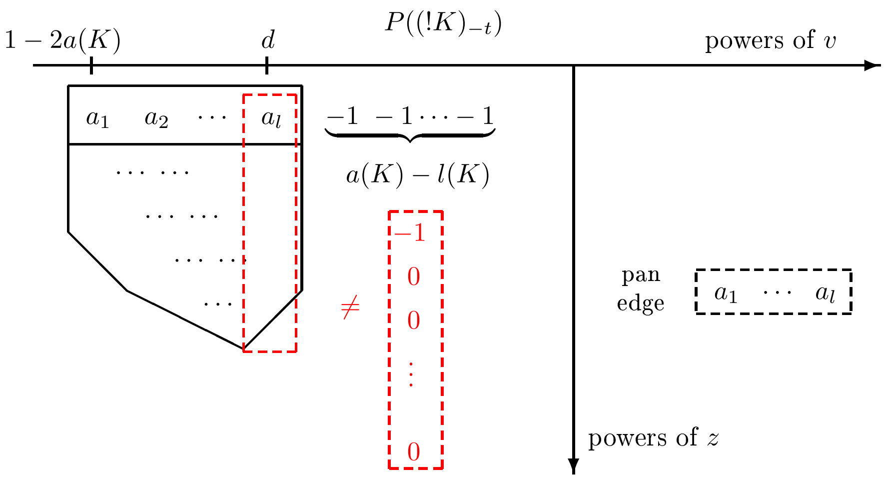
$$0 > d \ge -1 - 2a(K) + 2l(K) \tag{5.62}$$

[2]We emphasize that the polynomials in this and the next example are not HOMFLY-PT polynomials of real knotted annuli, i.e., the reader should not try to guess what K they were obtained from; we just hand-invented the polynomials for illustrative purposes.

so that

$$[P((!K)_{-t})]_{v^d} \neq -z \tag{5.63}$$

holds. (The condition (5.15) mirrors through (5.61) to (5.17).)



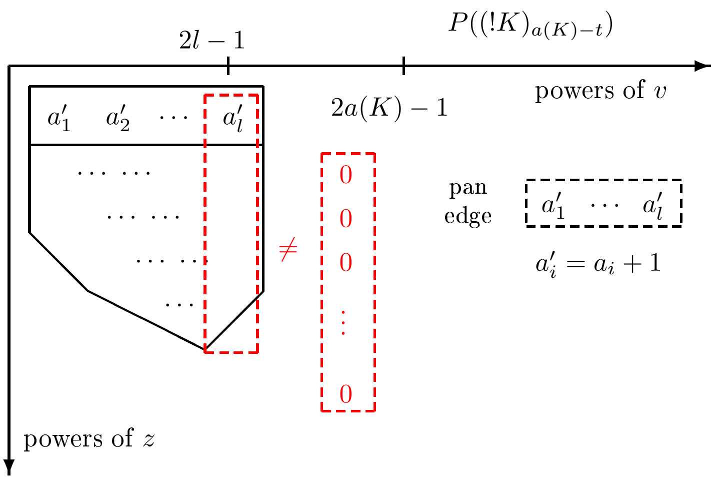
(5.64)

The repeated application of (5.12) then shows

$$\min \deg_v P((!K)_{a(K)-t}) \geq 1$$

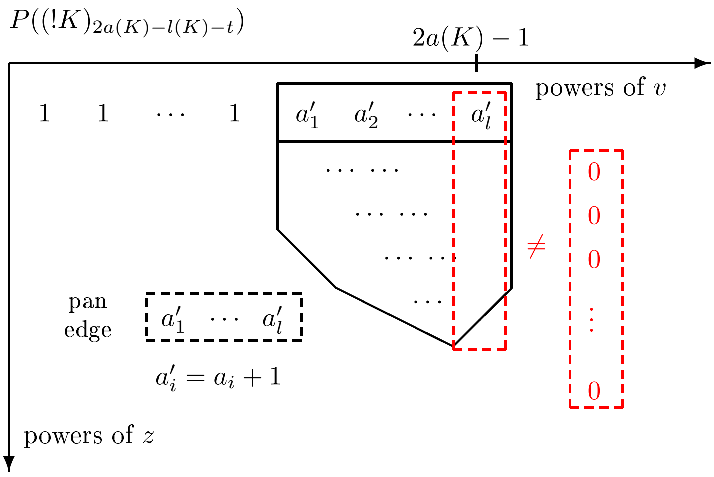
and by (5.62)

$$\max \deg_v P((!K)_{a(K)-t}) \geq 2l(K) - 1\,. \tag{5.65}$$



To see this last inequality (5.65), note that the terms annihilated by (5.12) when t increases are exactly those for $d < 0$ where (5.63) does not hold. Since $a(K) = a(!K)$, the inequality (5.65) means that the largest t' with $\max \deg_v P((!K)_{t'}) \leq 2a(!K) - 1$ satisfies

$$t' \leq 2a(K) - l(K) - t\,.$$



Now we can apply Lemma 5.1 on $!K$. We have

$$\lambda(!K) \le t' \le 2a(K) - l(K) - t = 2a(K) - l(K) - \lambda_{min}(K) \le 2a(K) - l(K) - \lambda(K)\,,$$

as we claimed. □

Example 5.37 We show a (fictitious) exemplary transformation of the $[P(K_t)]_{z^1}$ terms with increasing t, with the symbolics used in (5.26).

$$\boxed{5\ 4\ 1}\ -1\ \ -1\ \Big|\ \to\ 5\ 4\ 1\ -1\ \Big|\ \to\ 5\ 4\ 1\ \Big|\ \to \tag{5.66}$$
$$\to\ 5\ 4\ \Big|\ 2\ \to\ 5\ \Big|\ 5\ 2\ \to\ \Big|\ \boxed{6\ 5\ 2}\ \to\ \Big|\ 1\ 6\ 5\ 2\ \to\ \Big|\ 1\ 1\ 6\ 5\ 2$$

It consists of 7 steps: $a(K) = 5$, $l(K) = 3$, thus $2a(K) - l(K) = 7$.

Remark 5.38 Matsuda [Ma] (see also [Ng]) proved

$$a(K) \ge \lambda(K) + \lambda(!K), \tag{5.67}$$

which improves the right inequality in (5.60). But in fact, Theorem 4.10 with Corollary 3.8 shows that equality holds, answering [Ng, Question 1]:

$$a(K) = \lambda(K) + \lambda(!K)\,. \tag{5.68}$$

Then Proposition 5.36 can be interpreted by saying how much the HOMFLY-PT polynomial "sees" from that geometric reasoning. But we approach (5.60) from the viewpoint of strong quasipositivity, which can later be adapted to quasipositivity. To make clear that even with Theorem 4.10, our argument is not redundant, we quote the statement here, albeit its treatise has to be moved out to [St].

Proposition 5.39 ([St]) With

$$\lambda_q(K) := \min\{\ t:\ A(K,t) \text{ is quasipositive }\}\,, \tag{5.69}$$

we have

$$\begin{cases} l(K) \le \lambda_q(K) + \lambda_q(!K) \le 2a(K) - l(K) & \text{if } K \text{ is not slice} \\ l(K) - 1 \le \lambda_q(K) + \lambda_q(!K) \le 2a(K) - l(K) + 1 & \text{if } K \text{ is slice} \end{cases} \tag{5.70}$$

Remark 5.40 When K is an amphicheiral knot, $K = !K$, then $A(K, 0)$ is an (orientedly) amphicheiral link. One can use this and (2.15) to conclude that in that case both $l'(K)$ (see Lemma 5.6) and $l(K)$ are even (see also (5.75)). This is compatible with the fact that $a(K)$ is even through (5.68). Furthermore, the a_i in (5.56) exhibit a shifted antisymmetry: in the normalization $d_{min} > 0$, they satisfy $a_i + a_{l(K)+1-i} = 1$.

For computational purposes, we repeat here the formal self-contained (but not very pleasant) expression for $l(K)$ and the estimate (5.58) that is valid for arbitrary t. Take $P = P(K_t)$ for some $t \in \mathbb{Z}$. The quantities d_{min} and d_{max} can be determined as follows. Set

$$\widetilde{\min\deg_v} P = \begin{cases} \min\deg_v P & \min\deg_v P < 0 \\ \min\{\, d > 0 \,:\, [P]_{v^d} \neq z \,\} & \min\deg_v P > 0 \end{cases}$$

and

$$\widetilde{\max\deg_v} P = \begin{cases} \max\deg_v P & \max\deg_v P > 0 \\ \max\{\, d < 0 \,:\, [P]_{v^d} \neq -z \,\} & \max\deg_v P < 0 \end{cases} .$$

Then

$$l(K) = \frac{1}{2}\left(\widetilde{\max\deg_v} P - \widetilde{\min\deg_v} P\right) + 1\,, \tag{5.71}$$

and setting

$$\theta(K) = [P]_{z^1}(v=1) + \left\{ \begin{array}{ll} \lceil -1/2 \min\deg_v P \rceil & \min\deg_v P < 0 \\ -\left\lfloor 1/2 \widetilde{\min\deg_v} P \right\rfloor & \min\deg_v P > 0 \end{array} \right\}, \tag{5.72}$$

(5.58) can be stated as

$$\theta(K) \,\leq\, \lambda(K) \,\leq\, \theta(K) + a(K) - l(K)\,. \tag{5.73}$$

Remark 5.41 Again, if (5.34) occurs, then one can adapt the arguments in Remark 5.28 to disambiguate the value for $\lambda(K)$. This gives a practical way to calculate this number for any given K.

The formula (5.72) can be also applied to

$$P = P_d(K_t) = P(K_t)|_{z^{\leq d}}$$

for any $d \geq 1$ (odd). This will give lower bounds for $\theta(K)$ that can in certain cases, in combination with (3.15) and (5.68), be used to determine $a(K)$ and $\lambda(K)$ when (5.11) is strict without calculating the entire $P(K_t)$.

Example 5.42 The knot $K = 11_{404}$ has arc index 10. But $\mathrm{MB}(K) = 9$. However, we know from (3.15) additionally that $\lambda(K) \geq 7$, $\lambda(!K) \geq 2$. But when we use the left inequality in (5.73) for $P = P_1(K_1)$, we get $\lambda(K) \geq 6$, and by taking the mirror image via (5.61), we obtain from (5.73) also $\lambda(!K) \geq 3$. For reference, the truncation $P_1((!K)_{-1})$ is given below (with the way of reading it as explained in §2.4).

```
44 404        -1        1
  -3    7                          9    -33     52    -44     20     -4
  -7    9          4       -4     22   -133    278   -282    124      6    -16
```

In this example $[P_1]_{z^1}(v=1) = -1$, and the braced term in (5.72) evaluates to 4, thus giving $\theta(!K) \geq 3$. Then we have

$$\lambda(K) \geq 7,\;\; \lambda(!K) \geq 3\,, \tag{5.74}$$

and with a size-10 grid of K, we have from (5.68) that $a(K) = 10$, and that (5.74) are also equalities.

Of course these conclusions would follow from computing $l(K) = 10$ as well, but the point is that $P_1(K_t)$ was about 17 times faster to obtain than the entire $P(K_t)$. (The difference is here between 0.03 and 0.5s CPU time, but more crossings will stretch the delays far less pleasantly.) A similar example is 11_{453}.

It may be very hard, though, to find *l-unsharp* examples where this method is effective. I.e., it should decide $a(K)$ when *both* $\mathrm{MB}(K)$ and $l(K)$ fail separately. This is related to the difficulty of Question 5.14, since one can see that

$$\theta(K) + \theta(!K) = l(K)\,. \tag{5.75}$$

This is our own "estimate" version of (5.68), given that the right hand-side of (3.15) satisfies a similar property for MB(K) instead of $l(K)$. Thus Question 5.14 can be extended here: is

$$\theta(K) \geq 1 - \min\deg_a F(K)\,? \tag{5.76}$$

The following is the refinement of Corollary 5.19.

Proposition 5.43 Assume the premise of Corollary 5.19 is satisfied for $i = 1$. Then (5.76) holds for K.

Proof. Assume first (5.42) holds. If $\operatorname{span}_a F(K) \geq 1$, then find an f with (5.40).

Write $\theta_{\text{ mod } 2}(K)$ for (5.72) when for $P = P(K_f)$, the degrees $\min\deg_v P$ and $\max\deg_v P$ are replaced by $\min\deg_v P_{\text{mod } 2}$ and $\max\deg_v P_{\text{mod } 2}$ for $P_{\text{mod } 2}(K_f)$, *provided* the analogue of (5.41) holds,

$$\min\deg_v P_{\text{mod } 2}(K_f) < 0 < \max\deg_v P_{\text{mod } 2}(K_f)\,. \tag{5.77}$$

(Since we need $[P(K_f)]_{z^1}(v = 1) = f$, we cannot entirely replace $P(K_f)$ by $P_{\text{mod } 2}(K_f)$ when we define $\theta_{\text{ mod } 2}(K)$.)

Then clearly

$$\theta_{\text{ mod } 2}(K) \leq \theta(K)\,. \tag{5.78}$$

Notice that by (5.40), the condition (5.77) is true. It is thus enough to show $\theta_{\text{ mod } 2}(K) = 1 - \min\deg_a F(K)$.

Furthermore,

$$\min\deg_v P_{\text{mod } 2}(K_f) = \min\deg_v (R_f)_{\text{ mod } 2}(K) = 2\min\deg_a(F) - 1 + 2f < 0\,. \tag{5.79}$$

The first equality is Rudolph's congruence, provided the value is negative, which it is by (5.77), and the second follows similarly to the equality in (5.43).

Thus (with $[P]_{z^1}(v = 1) = f$; keep in mind (5.55))

$$\theta_{\text{ mod } 2}(K) = f + \lceil -1/2 \min\deg_v P_{\text{mod } 2}\rceil = \lceil -1/2\,(2\min\deg_a(F) - 1)\rceil = 1 - \min\deg_a F(K). \tag{5.80}$$

With (5.78) we are done.

When (5.42) holds and $\operatorname{span}_a F(K) = 0$ (i.e., $F(K) = 1$ by Remark 5.20), then notice that in the preceding argument, we only needed the left inequality in (5.40) (in conjunction with (5.79)) to evaluate $\theta_{\text{ mod } 2}(K)$ as in (5.80).

But for the left inequality in (5.40) alone, we do not need $\operatorname{span}_a F(K) \geq 1$ to find such an f. Then the rest of the argument repeats.

This same observation allows us to relax (5.42) to the stated assumption. The fact that we do not need the right inequality in (5.77) applies for the case that $\operatorname{span}_a F(K) \geq 1$ as well. □

Computation 5.44 The inequality (5.76), and hence also (5.35), is true for prime knots K up to 16 crossings, as was verified by refining and extending Computation 5.21. Here we need to treat mirror images separately. The premise of Corollary 5.19 for $i = 2$ will yield (5.76) for $!K$. (The verification for 16 crossings took about 33h with about the same computing capacity.)

Because of (5.75), Proposition 5.17 implies that (5.76) must hold as an equality when K is alternating. There is, though, one further noteworthy special case to add besides alternating knots.

Corollary 5.45 For every positive knot K, the inequality (5.76) holds as an equality.

Proof. By Yokota's result [Yo], $\min\deg_a F(K) = \min\deg_v P(K) = 2g(K)$, and up to variable change, the minimal terms coincide. And by (2.16), the term $[P(K)]_{v^{2g(K)}z^{2g(K)}} = \pm 1$ is odd. This shows (5.76).

We recall that by the argument of Tanaka [Ta2], for a positive knot K, we have

$$\lambda(K) = 1 - \min\deg_a F(K)\,, \tag{5.81}$$

and thus (5.76) is an equality. □

6 Braid indices revisited (and problematized)

6.1 Framing cones and the arc index

Here we summarize some remarks provided on various braid indices, and add discussion of related natural questions. They are meant to point out a series of subtleties, which may be significant or not, but which are easy to overlook while less straightforward to resolve. One having some particular importance in this context is Question 4.6. We reformulate part (a) here as a conjecture, with the insight gained from Corollary 4.3 and Remark 5.24.

Conjecture 6.1

$$a(K) = \min_{t\in\mathbb{Z}} b(A(K,t)) \tag{6.1}$$

The following reasoning will appear in several modified versions below, thus we record it as a lemma. Compare with Theorem 4.10.

Lemma 6.2 Assume (6.1) is true. Then (3.8) holds, in particular λ_{min} is unique.

Proof. Take an $a(K)$-band positive band presentation of $A(K,t)$ for $t = \lambda_{min} \geq \lambda(K)$, and make one band negative. By Remark 3.2, one has then an $a(K)$-band presentation of $A(K, t-1)$. Now since $A(K,t)$ is strongly quasipositive, it is Bennequin-sharp. But

$$\chi(A(K,t)) = \chi(A(K,t-1))\,, \tag{6.2}$$

and thus the $a(K)$-band presentation of $A(K,t-1)$ is not Bennequin-sharp, i.e., it does not make (2.6) an equality. But still $b(A(K,t-1)) = a(K)$ by (6.1). Now, if $A(K,t-1)$ is strongly quasipositive, then because of Theorem 2.2, every minimal braid representative of $b(A(K,t-1))$ would make (2.6) an equality. Thus we have that $A(K,t-1)$ is not strongly quasipositive. This means that $t-1 < \lambda(K)$, and so $t \leq \lambda(K)$, with the reverse inequality already observed. □

Remark 6.3 Note that Conjecture 6.1, when K is alternating, is related to Proposition 5.17. But it is not entirely implied by it, because of the sporadic collapsing scenario elucidated in the proof of Theorem 5.7. The way $l(K)$ was defined, $\mathrm{MFW}(K_t) < l(K)$ for some t can occur. Of course, replacing $l(K)$ with the bound $l'(K)$ in (5.10) avoids the collapsing problem. But we remind from the proof of Theorem 5.7 that we verified (5.10) to be (even very) unsharp in same cases.

More generally than (3.8), we have:

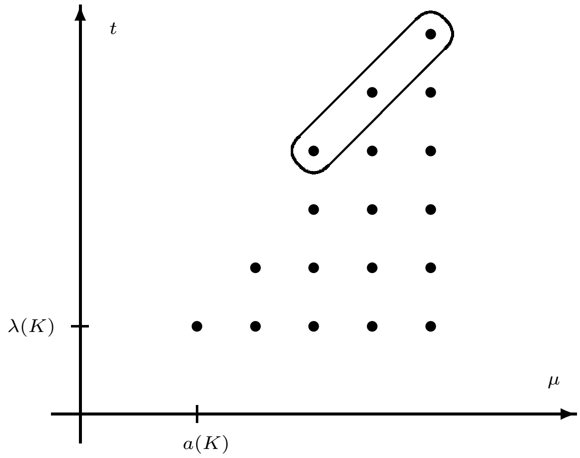
Lemma 6.4 Conjecture 6.1 implies a positive answer to Question 4.8, that $\Phi(K)$ is a single cone

$$\Phi(K) = C(a(K), \lambda(K))\,.$$

Proof. Conjecture 6.1 implies that in any band presentation on $s = a(K) + k$ strings with $> k$ negative bands will give an non-strongly quasipositive $A(K,t)$. The framing t changes with the sign of bands in an obvious way (compare with Remark 3.2). Thus if $(s,t) \in \Phi(K)$, then $t - (s - a(K)) < \lambda(K)$, in particular $(s, t-(s-a(K))) \notin \Phi(K)$. Therefore,

$$(s,t) \in \Phi(K) \Longrightarrow t \leq \lambda(K) + s - a(K)\,.$$

That is, there are no points in $\Phi(K)$ like the encircled:



This shows the cone shape of $\Phi(K)$. □

Lemma 6.4 pertains to the situation one may expect. But one can also use Theorem 2.2 for a version when Conjecture 6.1 is unresolved (or false).

Definition 6.5 Define the *defect* of K by

$$\delta(K) = a(K) - \min_{t \in \mathbb{Z}} b(A(K,t))$$

Then the argument for Lemma 6.4 modifies to show that an $a(K)$-band positive band presentation of $A(K,t)$ gives

$$\lambda(K) \leq t \leq \lambda(K) + \delta(K)\,, \tag{6.3}$$

and any positive band presentation of $A(K,t)$ on $s = a(K) + k$ strings will have

$$\lambda(K) \leq t \leq \lambda(K) + \delta(K) + k = \lambda(K) + \delta(K) + s - a(K)\,. \tag{6.4}$$

From this, we can conclude the following.

Proposition 6.6 For a non-trivial knot K, we have that $\Phi(K)$ is the union of at most $1 + \delta(K)$ cones.

Note that for $K = \bigcirc$, we have $\delta(K) = 0$, so that the claim is false due to the circumstance (3.7). (But, again, this case can be worked out separately: see Example 4.9.) In Remark 5.24 we have verified that $\delta(K) = 0$ for all prime knots K up to 10 crossings.

Proof. The condition (6.4) places (s,t) into a trapezoid which is the union of the cones $(a(K),t)$ for t in (6.3). Now, $\Phi(K)$ in obviously only contained in this union. Call a cone $C(\mu,t)$ in $\Phi(K)$ *essential*, if it is not properly contained in any other cone in $\Phi(K)$. Among cones $C(\mu,t)$ of fixed $t-\mu$ in $\Phi(K)$, there is always a maximal one, namely the one of the smallest μ. The same is true among cones $C(\mu,t)$ of fixed t in $\Phi(K)$. Note also that there are no values t with $\lambda(K) \leq t < \lambda_{min}$, since for $K \neq \bigcirc$, we have

$$\lambda_{min} = \lambda(K)$$

by Theorem 4.10.

Also, for each value $x = \lambda(K) + 1 - a(K), \ldots, \lambda(K) + \delta(K) - a(K)$ there is at most one essential cone $C(\mu,t)$ in $\Phi(K)$ with $t - \mu = x$. We call this essential cone *type* X. Obviously $C(a(K),\lambda(K))$ is also essential, and every other essential cone is of type X, by the above maximality remark. Now we have at most $\delta(K)$ type X essential cones. With $C(a(K),\lambda(K))$, this completes a set of $\delta(K)+1$ essential cones, as claimed. □

Obviously, from the definition,

$$\delta(K) \le a(K) - 2b(K)\,.$$

Thus in particular from (6.4), we have

$$\lambda(K) \le t \le \lambda(K) + s - 2b(K)$$

for any positive band presentation of $A(K,t)$ on $s \ge a(K)$ strings. Note also that, for computational purposes, one may replace '$1+\delta(K)$' in Proposition 6.6 by '$1+a(K)-l(K)$', with an analogous proof argument. (An analogous caveat regarding $K = \bigcirc$ is needed, where $a(K)=l(K)=2$; see (5.19).) We thus obtain the following proposition.

Proposition 6.7 When K is a non-trivial knot, then $\Phi(K)$ is the union of at most $1+a(K)-l(K)$ cones. □

6.2 Indices from braided surfaces

We return to Definition 2.1, and the inequality

$$b_{sqp}(S) \ge b(S)$$

for a strongly quasipositive surface S.

Question 6.8 While it is more than suggestive, we do not know if always equality holds. I.e., is every strongly quasipositive surface always realizable on its minimal number of strings in a positive band presentation?

Because of Theorem 2.2, this is true if $b(S)=b(K)$ (where of course $K=\partial S$). This is also related to the Baker-Motegi question if all minimal genus surfaces of a strongly quasipositive knot K are strongly quasipositive (see [St2]). From [HS], we know that $b(S)>b(K)$ for some minimal genus surface S of K. But S (and K) is not strongly quasipositive in these examples. Rudolph's question (4.1) is then equivalent to asking whether

$$b_{sqp}(S) = b(K) \tag{6.5}$$

is satisfied for *some* strongly quasipositive surface S of K. It is tempting to ask if (6.5) holds in fact for *every* strongly quasipositive surface S of K.

In case of the links $L=A(K,t)$ and $W_\pm(K,t)$, the minimal genus surfaces S_L of L are unique (and plumbing equivalent), so there is no need to distinguish between $b_b(S_L)$ and $b_b(L)$, and between $b_{sqp}(S_L)$ and $b_{sqp}(L)$.

Proposition 6.9 We obviously have

$$\min_{t\ge\lambda(K)} b_{sqp}(A(K,t)) = a(K)\,, \tag{6.6}$$

and for $t\ge\lambda(K)$, we can incorporate Whitehead doubles into the diagram as

$$\begin{array}{ccc} b_{sqp}(A(K,t)) & \ge & b(A(K,t)) \\ (*)\ \ \mathrel{\rotatebox{90}{$\le$}} & & \\ b_{sqp}(W_+(K,t)) & \ge & b(W_+(K,t)) \end{array} \tag{6.7}$$

Also, if K is l-sharp, then all inequalities are equalities.

Proof. The vertical inequality (*) holds because one can double any (positive) band in a strongly quasipositive band presentation of a t-twisted annulus for K (Example 3.13).

Now, consider the case that $l(K) = a(K)$. Since for $K = \bigcirc$ the equality questions in (6.7) can be settled by direct inspection, assume that $K \neq \bigcirc$, to avoid complications.

Consider $L = A(K, \lambda(K))$. We have

$$\max \deg_v P(L) = 2a(K) - 1\,, \tag{6.8}$$

and this means by (2.17) that an $a(K)$-braid (band) presentation of L cannot be of writhe less than $a(K)$. Since we did not assume $l'(K) = a(K)$, there may be a cancellation of terms in z-degree 1 (similarly to the first polynomial in Table 1). Thus $\min \deg_v P(L) > 1$ is, in principle, possible. But the writhe of an $a(K)$-braid (band) presentation of L cannot be more than $a(K)$ due to Bennequin's inequality (2.6). This means that the writhe of an $a(K)$-braid (band) presentation of L is unique, and hence $b(L) = a(K)$.

Then one can start with $t = \lambda(K)$ and propagate the bound in (2.17) through the recursion (5.4), while applying positive stabilizations (see (3.9)). □

Remark 6.10 By noting that we needed in the above proof only (6.8), for which $l(K) = a(K)$ is sufficient but not necessary, one also obtains equalities in (6.7) for $K = 10_{132}$. Pictorially speaking, this extra argument succeeds because the "missing terms" in $P(A(K,t))$, accounting for the difference (5.34), are missing "at the bottom", i.e., in low v-degrees. (See the first polynomial in Table 1.) The mirroring of 10_{132} of course continues to be relevant; keep in mind Example 3.7. And the situation immediately changes when v-conjugating the polynomial (by (2.15)), which explains why the trick definitely fails for the mirror image $!10_{132}$.

It follows from Computation 5.23 that all inequalities in (6.7) are equalities at least when minimum over $t \geq \lambda(K)$ is taken. This then holds for all Rolfsen knots, with mirror images (see also Example 6.12).

We can expect in (6.7) the horizontal '$\geq$' to be '$=$' in general, in accordance with Rudolph's Question (4.1). However, we do not know about (*). Obviously $S_{W_+(K,t)} = S_{A(K,t)} * H$ is a plumbing with a positive Hopf band H. But we know that

$$b_{sqp}(S * H) < b_{sqp}(S)$$

is possible, even for a strongly quasipositive fiber (in particular unique minimal genus) surface S; examples were given in [St2]. These examples, unsurprisingly, have higher genus, but they should still caution about seeing (*) as suggestive in some way.

Also, regarding (6.6), we can add

$$\min_{t \geq \lambda(K)} b_{sqp}(A(K,t)) = a(K) = \min_{t \in \mathbb{Z}} b_b(A(K,t))\,, \tag{6.9}$$

because every band presentation of Bennequin surface of $A(K,t)$ gives a grid diagram of K, and gives a strongly quasipositive surface of $A(K,t')$ for some $t' \geq \lambda(K)$ by making all bands positive.

Proposition 6.11 Then for instance for $t < \lambda(K)$, we have a similar diagram of inequalities to (6.7)

$$\begin{array}{ccc} b_b(A(K,t)) & \geq & b(A(K,t)) \\ (**)\ \ \rotatebox{90}{$\leq$} & & \\ b_b(W_-(K,t)) & \geq & b(W_-(K,t)) \end{array} \tag{6.10}$$

And if K is alternating, then all inequalities are equalities.

Proof. The inequality (**) results from doubling a negative band in a minimal band presentation (a negative band always exists when $t < \lambda(K)$; see the remarks following Example 3.13.

For the rest of the proof, we assume $l(K) = a(K)$, and argue that all inequalities are in fact qualities. We can infer this with a similar thought to Proposition 6.9. (Again, exclude $K = \bigcirc$ after a direct check.)

First, when $t \geq \lambda(K)$ (and K is l-sharp), then $A(K,t)$ has a minimal string band presentation that is positive. The argument with Bennequin's inequality is still needed to ascertain $b(A(K,\lambda(K))) = a(K)$ if

$$\mathrm{MFW}(A(K,\lambda(K))) < a(K)\,. \tag{6.11}$$

By Theorem 2.2, then any other minimal string minimal genus (i.e., Bennequin) band presentation of $A(K,t)$ must be positive either. The same holds for $W_+(K,t)$. Thus we can fall back onto Proposition 6.9. Simiarly, when $t \leq \lambda(K) - a(K)$, then any minimal string mand presentation of $A(K,t)$ must be negative, and we can use Proposition 6.9 for the mirrored links. Thus we assume now thourhout the rest of the proof that

$$\lambda(K) - a(K) < t < \lambda(K)\,, \tag{6.12}$$

and argue that all quantities in (6.10) are equal to $a(K)$. We know

$$b_b(W_-(K,t)), b_b(A(K,t)) \leq a(K)\,, \tag{6.13}$$

by construction of the band presentation (use Nutt's construction on a minimal grid diagram of K and make some bands negative).

The only framing t for which cancellation (6.11) may collapse the bound $\mathrm{MFW}(A(K,t)) < a(K)$ is when all bands in an $a(K)$-strand band presentation of $A(K,t)$ are positive or negative, that is, $t = \lambda(K)$ or $t = \lambda(K) - a(K)$. This case was excluded with (6.12). Thus $b_b(A(K,t)) = b(A(K,t))$.

However, the situation for $b_b(W_-(K,t)) \geq b(W_-(K,t))$ is slightly trickier, since a cancellation may occur for $t = \lambda(K) - 1$. (Indeed, as we will see from long computations in [St], there are *very* difficult l-sharp examples K, among others, $(p,3,-3)$-pretzel knots.)

However, under the stronger (keep in mind Proposition 5.17) restriction that K is alternating, it can be seen, essentially because $k_1, k_2 \geq 1$ in (5.39), that this cancellation never occurs. So, in that case as well

$$\mathrm{MFW}(W_-(K,t)) = \mathrm{MFW}(A(K,t)) = a(K)\,. \tag{6.14}$$

Finally, to see (**) is an equality, it is enough to see that $b(A(K,t)) = b(W_-(K,t))$. We can restrict to the values (6.12). Then we know (6.14) and (6.13). This is sufficient. □

Again (while it is tempting to suspect) we do not know if equalities hold in general.

Example 6.12 From Computation 5.23, we know that for all $t \in \mathbb{Z}$,

$$b(A(10_{132},t)), b(W_\pm(10_{132},t)) \geq 9 = a(10_{132})\,. \tag{6.15}$$

Obviously, as in Table 1, is it possible to write down explicit band presentations of $A(10_{132},t)$ and $W_-(10_{132},t)$ for *some* $t < \lambda(10_{132})$ on 9 strings, so that we have

$$b_b(A(10_{132},t)), b_b(W_-(10_{132},t)) \leq 9\,.$$

With Computation 5.23 we again know that thus for $K = 10_{132}$, the inequalities (6.10) are equalities *at least when their hand sides are minimized over* $t < \lambda(K)$. Under mirroring (using the computations and band presentations for $W_+(10_{132},t)$), we can conclude the same for $K = !10_{132}$, and thus for all Rolfsen knots.

When (5.34) occurs, though, this reasoning always relies on an explicit check for specific t using a 2-cable polynomial. And while we expect l-unsharp knots to be relatively rare, such instances K clearly increase with crossing number (see Example 5.12). The method in Computation 5.23 soon becomes problematic complexity-wise, despite algorithmic optimizations. This puts a limit to the capacity of our algebraic approach to tackle a geometric issue like the sharpness of the inequalities (6.10). (But of course it is the only information we have available so far.)

As an application of Propositions 6.9 and 6.11, we have one of our statements of the introduction, formulated there for more self-containedness (only) for alternating knots via Proposition 5.17.

Corollary 6.13 Assume K is l-sharp.

1. Assume $L = A(K,t)$ for some t. Then L has a minimal string Bennequin surface. Also, if L is strongly quasipositive, then L has a minimal string strongly quasipositive band presentation.

2. Let t' be so that
$$\max\deg_v P(A(K,t')) > 0$$
and assume
$$\max\mathrm{cf}_{\,v} P(A(K,t')) \neq \pm z^{-1}\,. \tag{6.16}$$
Assume $L = W_+(K,t)$ for some t. Then L has a minimal string Bennequin surface. Also, if L is strongly quasipositive, then, *without (6.16)*, L has a minimal string strongly quasipositive band presentation.

3. Let t' be so that
$$\min\deg_v P(A(K,t')) < 0$$
and assume
$$\min\mathrm{cf}_{\,v} P(A(K,t')) \neq \pm z^{-1}\,. \tag{6.17}$$
Assume $L = W_-(K,t)$ for some t. Then L has a minimal string Bennequin surface.

Proof. If $K = \bigcirc$, then again the claims are easy to test explicitly, so let us exclude this case henceforth.

If $L = A(K,t)$ then Propositions 6.9 and 6.11 show the claim directly. (Note that in the case $L = A(K,t)$ and K is alternating, one can obtain this result from [DM] as well; see Remark 5.18.) They also do except for the Bennequin surface case when $L = W_+(K,t)$ and $t < \lambda(K)$ and $L = W_-(K,t)$ and $t \geq \lambda(K)$.

When $L = W_+(K,t)$ and $\lambda(K) - a(K) < t < \lambda(K)$, then a band presentation of $A(K,t)$ on $a(K)$ strands has a positive band, and one can double it to obtain L. It can be checked with a skein calculation that $b(A(K,t)) = \mathrm{MFW}(A(K,t)) = \mathrm{MFW}(W_+(K,t))$. Condition (6.16) is needed for $t = 1-\lambda(K)-a(K)$, but this case is not relevant for strong quasipositivity.

When $L = W_+(K,t)$ and $\lambda(K)-a(K) \geq t$, then grid-stabilize positively a band presentation on $\lambda(K)-t$ strands and double a positive band to obtain L. By a skein calculation $\mathrm{MFW}(W_+(K,t)) = \lambda(K) - t + 1$.

Finally, when $L = W_-(K,t)$, the claim follows from the argument for $W_+(K,t)$ under mirroring. (However, L is never strongly quasipositive, so this part trivializes.) We need (6.17) to handle the case $t = \lambda(K) - 1$.

When $L = W_-(K,t)$ and $t \geq \lambda(K)$, then grid-stabilize a $t - \lambda(K) + a(K)$-string band representation of $A(K,t)$ negatively and double a negative band. By a skein calculation $\mathrm{MFW}(W_-(K,t)) = t - \lambda(K) + a(K) + 1$. □

We finish by stating the generalization of [DM], which follows along the same lines as Corollary 6.13.

Proposition 6.14 Assume K is a non-trivial l-sharp knot. Then
$$b(A(K,t)) = \begin{cases} \lambda(K) - t & \text{if } t \leq \lambda(K) - a(K) \\ a(K) & \text{if } \lambda(K) - a(K) \leq t \leq \lambda(K) \\ t - \lambda(K) + a(K) & \text{if } t \geq \lambda(K) \end{cases} \tag{6.18}$$
□

Note that for K being the unknot this expression still holds when $\lambda(K) = 0$ is replaced by 1 (see (3.16)). There is, of course, also a formula for $b(W_\pm(K,t))$ for l-sharp K (excluding the exceptional t_0 for W_-, unless K is assumed alternating), but we leave this to the reader. Furthermore, comparison with Diao-Morton's statement implies, besides (5.38), an expression of $\lambda(K)$ for K alternating in terms of a checkerboard coloring of a reduced alternating diagram of K. (But one must reverse signs, as explained below Definition 3.1.) When one uses the degrees of F instead of the geometric quantities (as can be done for alternating knots), then this identification was given by Yokota [Yo2].

Example 6.15 The case of l-unsharp K is far more complicated. For instance, when $K = 10_{132}$ (and as a refinement of Remark 6.10), then in (6.18) we still know the last two alternatives. For $\lambda(10_{132})-a(10_{132}) \leq t \leq \lambda(10_{132})$, we have (6.15), and for $t \geq \lambda(10_{132})$, the trick with Bennequin's inequality can still be used. But this trick does not work for $t < \lambda(10_{132}) - a(10_{132})$, because the missing terms in $P(A(K,t))$ occur at the "wrong" end of the v-degree. Potentially one may adapt some consideration like for Proposition 5.9, but this only promises a tenuous and very involved argument.

For further applications to the Bennequin sharpness problem (2.9) of Whitehead doubles, see [St].

Acknowledgement. The second author (A.S.) had some helpful related discussions with and references from B. Ahn. We further thank the organizers of The First International Online Knot Theory Congress, in honor of Lou Kauffman's 80th anniversary, for making it a great event, and also for giving us the opportunity to present this research. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (2023R1A2C1003749).

References

[Ba] S. Baader, *Quasipositivity and homogeneity*, Math. Proc. Camb. Phil. Soc. **139(2)** (2005), 287–290.

[BP] Y. Bae and C.-Y. Park, *An upper bound of arc index of links*, Math. Proc. Cambridge Philos. Soc. **129** (2000), 491-500.

[Be] D. Bennequin, *Entrelacements et équations de Pfaff*, Soc. Math. de France, Astérisque **107-108** (1983), 87–161.

[BKL] J. S. Birman, K. Ko and S. J. Lee, *A new approach to the word and conjugacy problems in the braid groups*, Adv. Math. **139(2)** (1998), 322–353.

[BM] J. S. Birman and W. W. Menasco, *Studying knots via braids VI: A non-finiteness theorem*, Pacific J. Math. **156** (1992), 265–285.

[BM2] J. S. Birman and W. W. Menasco, *Studying links via closed braids II: On a theorem of Bennequin*, Topology Appl. **40(1)** (1991), 71–82.

[Bu] Benjamin A. Burton, *The next 350 million knots*, 36th International Symposium on Computational Geometry (SoCG 2020) (S. Cabello, D.Z. Chen, eds.), Leibniz Int. Proc. Inform., vol. **164**, Schloss Dachstuhl-Leibniz-Zentrum fuer Informatik, 2020, 25:1–25:17, `https://regina-normal.github.io/data.html`

[Cr] P. R. Cromwell, *Embedding knots and links in an open book. I. Basic properties*, Topology Appl. **64** (1995), 37–58.

[Cr3] P. R. Cromwell, *Arc presentations of knots and links*, from "Knot theory" (V. F. R. Jones, J. Kania-Bartoszynska, J. H. Przytycki, P. Traczyk, V. G. Turaev, editors), Banach Center Publ. **42**, Polish Acad. Sci. Inst. Math., Warsaw (1998), 57–64.

[DM] Y. Diao and H. Morton, *The braid indices of the reverse parallel links of alternating knots*, Algebr. Geom. Topol. **24** (2024), 2957–2970.

[DP] I. A. Dynnikov and M. V. Prasolov, *Bypasses for rectangular diagrams. A proof of the Jones conjecture and related questions*, Trans. Moscow Math. Soc. **2013**, 97–144, `doi.org/10.1090/S0077-1554-2014-00210-7`.

[EH] J. Etnyre and K. Honda, *On connected sums and Legendrian knots*, Advances in Mathematics **179(1)** (2003), 59–74.

[FLL] P. Feller, L. Lewark and A. Lobb, *Almost positive links are strongly quasipositive,* Math. Ann. **385(1-2)** (2023), 481–510.

[Fe] E. Ferrand, *On Legendrian knots and polynomial invariants*, Proc. Amer. Math. Soc. **130(4)** (2001), 1169–1176.

[FT] D. Fuchs and S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36(5)** (1997), 1025–1053.

[FW] J. Franks and R. Williams, *Braids and the Jones polynomial.* Trans. Amer. Math. Soc. **303(1)** (1987), 97–108.

[HS] M. Hirasawa and A. Stoimenov, *Examples of knots without minimal string Bennequin surfaces*, Asian Journal Math. **7(3)** (2003), 435–446.

[HT] J. Hoste and M. Thistlethwaite, *KnotScape*, a knot polynomial calculation and table access program, available at `http://www.math.utk.edu/~morwen`.

[IS] T. Ito and A. Stoimenov, *Invariants of weakly successively almost positive links*, preprint `arXiv.2208.10728`.

[J+] G. T. Jin, H. Kim, G.-S. Lee, J. H. Gong, H. Kim, H. Kim and S. A. Oh, *Prime knots with arc index up to 10*, in "Intelligence of Low Dimensional Topology 2006", Series Knots and Everything, Vol. **40** (World Scientific, Hackensack, NJ, 2007), 65–74.

[JLS] G. T. Jin, H. J. Lee and A. Stoimenov, *Grid diagrams, link indices, and strong quasipositivity*, available at request from the third author.

[JP] G. T. Jin and W. K. Park, *Prime knots with arc index up to 11 and an upper bound of arc index for non-alternating knots*, J. Knot Theory Ramifications **19** (2010), 1655–1672.

[J] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335–388.

[LaM] D. J. LaFountain and W. W. Menasco, *Embedded annuli and Jones' conjecture*, Algebr. Geom. Topol. **14(6)** (2014), 3589–3601.

[LT] H. J. Lee and H. Takioka, *On the arc index of cable links and Whitehead doubles*, Journal of Knot Theory and Its Ramifications **25(7)** (2016), 1650041.

[LiM] W. B. R. Lickorish and K. C. Millett, *A polynomial invariant for oriented links*, Topology **26(1)** (1987), 107–141.

[LvM] C. Livingston and A. Moore, *KnotInfo*, online knot invariant database, `https://knotinfo.org/`

[LvN] C. Livingston and S. Naik, *Ozsváth-Szabó and Rasmussen invariants of doubled knots*, Algebr. Geom. Topol. **6** (2006), 651–657.

[Ma] H. Matsuda, *Links in an open book decomposition and in the standard contact structure*, Proc. Amer. Math. Soc. **134(12)** (2006), 3697–3702.

[Mo] H. R. Morton, *Seifert circles and knot polynomials*, Proc. Camb. Phil. Soc. **99** (1986), 107–109.

[MB] H. R. Morton and E. Beltrami, *Arc index and the Kauffman polynomial*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 41–48.

[MS] H. R. Morton and H. B. Short, *The 2-variable polynomial of cable knots*, Math. Proc. Camb. Philos. Soc. **101** (1987), 267–278.

[Mu] K. Murasugi, *An estimate of the bridge index of links*, Kobe J. Math. **5(1)** (1988), 75–86.

[Ng] L. Ng, *On arc index and maximal Thurston-Bennequin number*, J. Knot Theory Ramifications **21(4)** (2012), 1250031, 11 pp.

[Nu] Ian J. Nutt, *Embedding knots and links in an open book III. On the braid index of satellite links*, Math. Proc. Camb. Phil. Soc. **126** (1999), 77–98.

[Oh] Y. Ohyama, *On the minimal crossing number and the braid index of links*, Canad. J. Math. **45(1)** (1993), 117–131.

[Ro] D. Rolfsen, *Knots and links*, Publish or Perish, 1976.

[Ru] L. Rudolph, *Braided surfaces and Seifert ribbons for closed braids,* Comment. Math. Helv. **58(1)** (1983), 1–37.

[Ru2] L. Rudolph, *Quasipositive plumbing (constructions of quasipositive knots and links. V)* Proc. Amer. Math. Soc. **126(1)** (1998), 257–267.

[Ru3] L. Rudolph, *An obstruction to sliceness via contact geometry and "classical" gauge theory*, Invent. Math. **119(1)** (1995), 155–163.

[Ru4] L. Rudolph, *Knot theory of complex plane curves*, in *Handbook of Knot Theory*, W. Menasco and M. Thistlethwaite (eds.), Elsevier Science, 2005, 329–428.

[Ru5] L. Rudolph, *A congruence between link polynomials*, Math. Proc. Cambridge Philos. Soc. **107** (1990), 319–327.

[St] A. Stoimenov, *Grid diagrams, link indices, and quasipositivity*, available at request from the author.

[St2] A. Stoimenov, *Realizing strongly quasipositive links and Bennequin surfaces,*, to appear in Publ. Res. Inst. Math. Sci., `http://www.stoimenov.net/stoimeno/homepage/papers.html`

[St3] A. Stoimenov, *Tabulating and distinguishing mutants*, Internat. Jour. Algebra Comput. **20(4)** (2010), 525–559, DOI: 10.1142/S0218196710005789

[St4] A. Stoimenov, *Knot data tables*, `http://www.stoimenov.net/stoimeno/homepage/ptab/`

[St5] A. Stoimenov, *On the crossing number of positive knots and braids and braid index criteria of Jones and Morton-Williams-Franks*, Trans. Amer. Math. Soc. **354(10)** (2002), 3927–3954.

[Ta] T. Tanaka, *Maximal Thurston-Bennequin numbers of alternating links*, Topology and its Applications **153(14)** (2006), 2476–2483.

[Ta2] T. Tanaka, *Maximal Bennequin numbers and Kauffman polynomials of positive links*, Proc. Amer. Math. Soc. **127(11)** (1999), 3427–3432.

[Th] M. B. Thistlethwaite, *On the Kauffman polynomial of an adequate link*, Invent. Math. **93** (1988), 285–296.

[To] Ichiro Torisu, *The additivity of the Thurston–Bennequin invariant of Legendrian knots*, Pacific J. Math. **210(2)** (2003), 359–365, `arxiv.org/abs/math/0103023`.

[Yo] Y. Yokota, *Polynomial invariants of positive links*, Topology **31(4)** (1992), 805–811.

[Yo2] Y. Yokota, *The Kauffman polynomial of alternating links*, Topology Appl. **65(3)** (1995), 229–236.