THE ALEXANDER POLYNOMIAL OF PLANAR EVEN VALENCE GRAPHS

This is a preprint. We would be grateful for any comments and corrections!

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Abstract. We show how the Alexander/Conway link polynomial occurs in the context of planar even valence graphs, refining the notion of the number of their spanning trees. Then we apply knot theory to deduce several statements about this graph polynomial, in particular estimates for its coefficients and relations between congruences of the number of vertices and number of spanning trees of the graph.

1. Introduction

A graph G will have for us possibly multiple edges and loop edges (edges connecting one and the same vertex). $V(G)$ will be the set of vertices of G, and $E(G)$ the set of edges of G (each multiple edge counting as a set of single edges). $v(G)$ and $e(G)$ will be the number of vertices and edges of G (thus counted), respectively.

We call a graph *even valence*, if all its vertices have even valence.

Consider an arbitrary even valence graph G. A *canonical* edge orientation is an orientation of the edges of G, such that one half of the edges incident to each vertex v in G are incomingly and outgoingly oriented w.r.t. v (compare to definition 9.4, p. 138, in [17]).

Let G be a connected even valence graph with one of its canonical edge orientations. Let $v \in V(G)$ be a root vertex.

Consider a spanning tree $\Gamma \subset G$. Γ has a *canonical* edge orientation "towards the root". This orientation is defined by requiring that each non-root vertex has exactly one outgoing edge in Γ, while the root has none. (The canonical edge orientation of Γ is not to be confused with the canonical edge orientation of G; since a tree is never an even valence graph, there is no overlap of the two notions.) We say that an edge e in Γ is *coherently* oriented, if its canonical orientation in Γ w.r.t. the root v is the same as its canonical orientation in G. Otherwise call e incoherently oriented. Define the *index* $i(\Gamma, G, v)$ of Γ in G to be the number of incoherently oriented edges in Γ.

Definition 1 Define a polynomial of G by

$$\Delta_G(t) = \Delta(G) := \sum_{\Gamma \subset G \text{ sp. tree}} t^{i(\Gamma,G,v)} \in \mathbb{Z}[t]\,. \tag{1}$$

We will often omit for simplicity the index or argument of Δ, when they are clear from the context. The abuse of v in the notation will become clear later.

Clearly $s(G) = \Delta_G(1)$ is the number of spanning trees of G. Beside this fact, one may wonder why it is interesting to consider this graph polynomial. However, in a special case this polynomial turns out to be well-known from knot

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theory, and (in this special case) latter can be applied in the study of it and in particular of the number of spanning trees of G. This relationship between graph and knot theory relies on early work of Kauffman [17] and the first author (see [27] for a review), and has proved fruitful in the study of graphs using knot theoretical methods (see e.g. [13]).

After stating some general features of Δ_G in the next section, we will explain in sections 3, 4 and 5 the relation of a special case of Δ_G to knot theory, and then state in §6 the properties of Δ_G we can prove using the tools it provides. We conclude describing in §7 some problems about Δ_G which would be of interest for knot theory and are made now (hopefully) more accessible by the graph theoretic description of Δ_G.

2. Some general properties of Δ_G

We start with the following presentation of Δ_G.

Theorem 1 Let G be an even valence graph with some fixed canonical edge orientation. Define a matrix $M = (m_{ij})$ of size $v(G) \times v(G)$ by

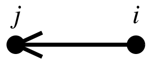
$$m_{ij} = \begin{cases} -\#\{\text{edges directed from vertex } i \text{ to vertex } j\} & i \neq j \\ {}^1\!/_2 \text{ valence of vertex } i & i = j \end{cases} .$$

Then

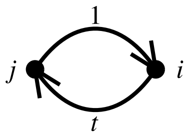
$$\Delta_G(t) = \det\left(\overline{M} + t\overline{M}^T\right),$$

where $\overline{M}$ means M with some row and column (of the same index) discarded.

Proof. This is an application of the Kirchhoff-Tutte matrix-tree theorem [40]. Explicitly, consider its most general version proved in [7, p. 379 bottom]. Replace each edge



in G by

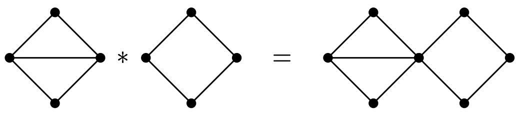


(2)

(note that the canonical edge orientation of a tree in (1) is opposite to the orientation of an arborescence there), and set $M_{ij} = 1$ or t as given by the labeling of the edges on the right of (2), and all $x_j = 1$. (See also [28, theorem p. 195 bottom]; further references are [5, 22, 44].) □

As a consequence we obtain some simple properties of Δ_G.

Definition 2 The *join* (or *block sum*, as called in [27]) '$*$' of two graphs is defined by



This operation depends on the choice of a vertex in each one of the graphs. We call this vertex the *join vertex*.

Definition 3 Let $[P]_{t^a} = [P]_a$ be the coefficient of t^a in a polynomial $P \in \mathbb{Z}[t^{\pm 1}]$. Let

$$\min\deg P = \min\{\, a \in \mathbb{Z} : [P]_a \neq 0 \,\}, \quad \max\deg P = \max\{\, a \in \mathbb{Z} : [P]_a \neq 0 \,\},$$
$$\operatorname{span} P = \max\deg P - \min\deg P, \quad \max\operatorname{cf} P = [P]_{\max\deg P}.$$

Proposition 1 Let G be an even valence graph with some fixed canonical edge orientation. Define Δ_G as above (for some fixed root vertex). Then

1) Δ_G is independent on the choice of root.
2) $\Delta_G(t) = t^{v(G)-1}\Delta_G(1/t)$, that is, Δ_G is invariant under switching simultaneously the orientation of all edges in G.
3) $\min\deg\Delta_G = 0$ and $\max\deg\Delta_G = v(G)-1$.
4) Δ_G is a positive polynomial, i.e. $[\Delta_G]_i > 0$ for any $0 \le i \le v(G)-1$.
5) For any two even valence graphs $G_{1,2}$ we have $\Delta_{G_1 * G_2} = \Delta_{G_1} \cdot \Delta_{G_2}$, in which way ever the join is performed.

Proof.

1) The first property follows from theorem 1, since M has zero entry sum in any row and column.
2) The second property is also straightforward from theorem 1. In particular,
$$\min\deg\Delta_G + \max\deg\Delta_G \,=\, v(G)-1\,. \tag{3}$$
3) Because of (3), for the third property it suffices to show that each G has a (spanning) tree of index 0 for *some* choice of root.

   For this we proceed by induction on $v(G)$. If $v(G) = 1$, then the claim is trivial.

   Otherwise consider G with $v(G) > 1$. Fix some edge e in G directed from vertex i to vertex j. Let G_e be G with e contracted and w be the vertex in G_e obtained by the unification of i and j. The contraction defines a map $\hat{e} : E(G)\setminus\{e\} \to E(G_e)$ given by replacing any vertex i and j occurring as source or destination of an edge in G by w. If we consider (and shall do so from now on) a multiple edge as a set of single edges, then $\hat{e}$ is a bijection. (Here a set is to be understood with the order of its elements ignored, but with their multiplicity counted, i.e. $\{1,1,2,3\} = \{1,1,3,2\} \neq \{1,2,3\}$.)

   G_e has by induction some index 0 tree (for some root). Then by the first part, there is an index 0 tree in G_e for any root, in particular for w. Let Γ be such a tree. Define a tree $\Gamma' \subset G$ by $\Gamma' = \hat{e}^{-1}(\Gamma)\cup e$. This gives an index 0 tree Γ' in G with root j, thereby completing the induction argument. (See lemma 9.5, p. 139 of [17] for a slightly different proof.)
4) For the fourth property one has to show that a tree of any possible index between 0 and $v(G)-1$ exists. This can be easily achieved by modifying the argument above for index 0, since by switching between vertices i and j as the new roots of Γ one can add e to Γ to be either a coherently or incoherently oriented edge in Γ'.
5) We already showed that Δ is independent on the choice of root, and the fifth property is straightforward to check for the root being the join vertex. □

Another property (whose importance will be motivated later) is as follows.

Proposition 2 For any even valence graph G with fixed canonical edge orientation and root v, and polynomial $\Delta = \Delta_G$ defined is in (1), we have $[\Delta]_0 \le [\Delta]_1$.

Proof. Let $\mathcal{T}_k$ be the set of rooted oriented spanning trees T of G of index k. Thus $|\mathcal{T}_k| = [\Delta]_k$. Now we define a map
$$\phi : \mathcal{T}_0 \to \mathcal{P}(\mathcal{T}_1)\setminus\{\varnothing\}\,,$$
where $\mathcal{P}(A)$ denotes the power set (set of all subsets) of A. We will have that
$$\phi(T_1)\cap\phi(T_2) = \varnothing\,, \tag{4}$$
for $T_1 \neq T_2$, so that the claim will follow.

For each vertex p of G fix a bijection χ_p between the incoming and outgoing edges of G w.r.t. p.

Take a tree $T \in \mathcal{T}_0$. Since T is a tree, T has *stumps*, i.e. vertices of valence one different from the root v. Take one of the stumps p, and let e be the outgoing edge w.r.t. p in T. Let $e' = \chi_p(e)$, and

$$T'(e) := (T \setminus e) \cup \chi_p(e).$$

We claim that $T' = T'(e) \in \mathcal{T}_1$. First $T'(e)$ is cycle-free. Otherwise, let C be a cycle in T'. If C does not contain e', then $T' \setminus e'$ contains C, and hence so does $T = (T' \setminus e') \cup e$, which is a tree, a contradiction. If C contains e', then it contains at least two edges incident to p in T, contradicting the assumption that p is a stump in T.

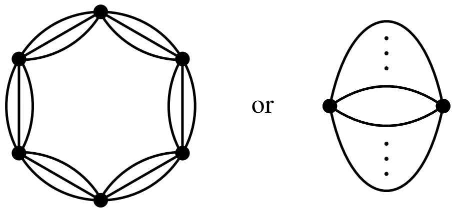
Thus T' is a forest. Clearly T' is a spanning forest, and as $e(T') = e(T) = v(T) - 1 = v(T') - 1$, it must be a spanning tree. Since e' is the only edge in T' oppositely oriented in G and T, we have $T'(e) \in \mathcal{T}_1$.

Then set

$$\phi(T) := \{T'(e) : e \text{ is incident to a stump in } T\}.$$

Since if $T' \in \phi(T)$, we have $T = (T' \setminus e') \cup \chi_p^{-1}(e')$, where e' is the (only) oppositely oriented edge in T' and p is the stump it is incident to in T'. This implies (4), and hence the assertion we wished to prove. □

Remark 1 Suppose $[\Delta]_0 = [\Delta]_1$. Then from the proof it follows that every tree in $\mathcal{T}_0$ must have exactly one stump. Therefore, G is a chain, in which edge is possibly replaced by the same number, say m, of parallel edges with the same orientation:



(2m edges).

In the first case,

$$\Delta_G(t) = m^{v(G)-1}\left(1 + t + \ldots + t^{v(G)-1}\right),$$

and in the second case $\Delta_G(t) = m(1+t)$.

3. Δ_G for planar graphs

A graph G is called *planar* if it is embeddable into the plane. It is *a priori* convenient to assume that a planar embedding of G is fixed. (Although we will later show that we can drop this assumption.) When a planar graph is equipped with a planar embedding, we call it a *plane* graph. Planar embeddings will be considered up to isotopy in the plane and change of the unbounded cell. (A *cell* is called a connected component of the complement of G in the plane.)

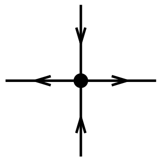
For plane graphs there is the classical notion of duality: to a plane graph G we associate its *dual* (plane) graph G^* by assigning to each cell of G a vertex of G^* and for each edge e an edge between the (vertices of) the regions E_1 and E_2 in G^*, where $e \subset \partial E_1$ and $e \subset \partial E_2$. Clearly $G^{**} = G$.

There is a bijection between bipartite and even valence graphs among plane graphs, given by duality. The fact should be folklore, but we give a proof, since it entails a construction (of alternating edge orientation) which will become of importance shortly.

Lemma 1 Let G be a plane graph. Then G is bipartite $\iff$ G^* is even valence.

Proof. If G is bipartite, then all cycles in G have even length. Therefore, the number of edges bounding a cell is even, and hence G^* is even valence. The converse follows from the next lemma: if G has even valence, then the edges of G can be alternatingly oriented. Then the edges bounding a fixed cell are all clockwise or all counterclockwise oriented as seen from inside this cell. The distinction between these two orientations gives the bipartition of G^*. □

Lemma 2 Let G be a plane bipartite graph. Then the edges of G can be oriented so that each two edges neighboredly incident to a vertex (incident to it and bounding a common cell) have different orientation (between incoming or outgoing) with respect to this vertex.



We call this an *alternating* edge orientation of G.

Proof. G can be simplified to the trivial graph (no edges) by removing cell boundaries. This does not spoil the even valence property. Thus it suffices to construct the edge orientation inductively. Whenever (boundary edges of) a new cell I are restored inside an old cell J, orient all edges of the boundary ∂I of I oppositely (between clockwise or counterclockwise) to those in ∂J. If there is no such J, then I connects two different connected components. In this case the orientation of I may be chosen properly after eventually reversing the orientation of all edges in one of the components. $\square$

Remark 2 The alternating edge orientation is unique up to reversing the orientation of all edges in each connected component.

It is clear that for a plane even valence graph G an alternating edge orientation is in particular a canonical edge orientation. Thus we can consider Δ_G defined as in (1) in this special case. This is the case related to knot theory. In order to explain this relationship, we need some knot theoretical preliminaries.

4. Checkerboard colorings and Alexander polynomial

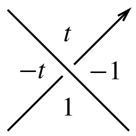
The aim of this section is to introduce one of the most fundamental invariants of knots and links, the Alexander polynomial [1] Δ (the coincidence of this notation with our graph polynomial is not accidental), and a way how this polynomial can be graph theoretically calculated.

4.1. A state model for the Alexander polynomial

We start by describing a state model for (the calculation of) the Alexander polynomial Δ developed by Kauffman from Alexander's original definition of Δ [1]. The ideas are described in Kauffman's book [17] (see in particular §§4,6 and 9), and, following his exposition [16], are presented here in a more detailed and slightly different form, also adapted to our subsequent applications to graphs. We start with a bit of terminology.

Definition 4 A *region* of a link diagram is a connected component of the complement of the (plane curve of) the diagram. An *edge* of D is the part of the plane curve of D between two crossings (clearly each edge bounds two regions). We call two regions *opposite* at a crossing p, if p lies in the boundary of both regions, but they do not share any of the four edges bounded by p.

Consider an oriented n crossing link diagram D and choose n regions $R_1,\dots,R_n$ in the complement of the diagram, such that the remaining two regions R'_1 and R'_2 are adjacent (that is, share an edge). Number the crossings of D to be $c_1,\dots,c_n$. If c_j is not adjacent to R_i (i.e. $c_j \notin \partial R_i$), then set $A_{i,j}=0$. Else consider the 4 regions around c_j and give each of them values of $A_{i,j}$ in ± 1, $\pm t$ depending on the side from which R_i meets c_j:



(5)

(the orientation of the undercrossing strand is irrelevant).

Then Alexander defines

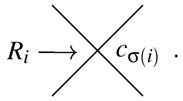
$$\Delta(D) \doteq \det(A_{i,j})_{i,j=1}^{n} ,$$

'$\doteq$' denoting equality up to units in $\mathbb{Z}[t,t^{-1}]$, and shows that (up to this multiplicative ambiguity) $\Delta(D)$ is independent on which diagram D of a given link L, and which adjacent regions $R'_{1,2}$ of D we choose, and thus becomes a link invariant Δ_L.

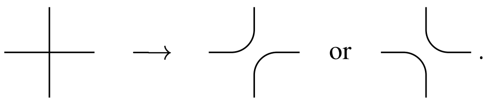
When writing

$$\det(A_{i,j}) = \sum_{\sigma\in S_n} (-1)^{\sigma} \prod_{i=1}^{n} A_{i,\sigma(i)} , \tag{6}$$

the contribution of a permutation σ is non-zero if and only if $R_i \mapsto c_{\sigma(i)}$ is an assignment of a crossing to a region meeting it, such that each crossing is assigned exactly once, and then this contribution is a monomial. Denote this correspondence by an arrow from the region to the crossing:



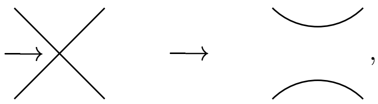
Consider $\hat{D} \subset \mathbb{R}^2$, the (image of) the associated immersed plane curve(s) of D. For each crossing (self-intersection) of $\hat{D}$ there are 2 ways to *splice* it:



(7)

We call a choice of splicing for each crossing a *state*. This terminology was introduced by Kauffman in the context of the bracket model for the Jones polynomial [15]. We call states for which the resulting collection of disjoint circles has only one component (a single circle) *monocyclic*.

Then, replacing in (7)



(8)

it is an easy exercise to see that these splicings of the crossings define a monocyclic state.

To see this, notice that if at some point the splicing (8) disconnects the diagram into 2 components $D_{1,2}$ of $n_{1,2}$ crossings, then the adjacency of the 2 regions $R'_{1,2}$ implies that for some $k \in \{1,2\}$, all regions of D_k except one are among the R_i's, such that σ must assign the n_k crossings of D_k to $n_k + 1$ regions, a contradiction.

Similarly, one argues that each monocyclic state can be realized only once, because a rearrangement of the arrows to give the same splicings will result in splicings which disconnect the diagram.

That each monocyclic state is indeed realized by a permutation, follows from considering the alternating diagram D' obtained from D by crossing changes.

A link diagram is called *alternating* if each strand alternatingly passes crossings as under- and overpass, and there is always a way to switch the crossings of any link diagram so as it to become alternating, canonical up to simultaneous switch of all crossings.

The number of monocyclic states of D is equal to the *determinant* of D' (see [19]); the determinant is the link invariant given by $\det(D') = |\Delta_{D'}(-1)|$. From this identity it follows, that each monocyclic state must indeed be realized by a permutation in (6), otherwise the number of (unit) monomials adding up in (6) to Δ_D, and hence to $\Delta_{D'}$, would be smaller than $\det(D') = |\Delta_{D'}(-1)|$, which is impossible.

Therefore, we have

Proposition 3 (see Theorem 2.4 of [17]) There is a bijection, given by (8), between arrow assignments with non-zero contributions to the sum in (6), and monocyclic states in the bracket model. □

Then the calculation of the determinant of the matrix $(A_{i,j})$ via (6) can be interpreted as a "state sum", the non-trivial summands being units and coming exactly from the monocyclic states in the bracket model.

Remark 3 A cancellation of the units contributed by such monocyclic states occurs iff the diagram D is non-alternating. This can be seen directly from the construction of $(A_{i,j})$, or by using the argument for the bracket model for the Jones polynomial V and the identity $\Delta(-1) = V(-1)$.

4.2. Checkerboard colorings and graphs

Here we describe the following construction, linking graph and knot theory (see e.g. [15]): given an alternating diagram D of a knot (or link; knots are considered one-component links), we can associate to it its checkerboard graph.

The *checkerboard coloring* of a link diagram is a map

$$\{\text{ regions of } D \text{ }\} \to \{\text{ black, white }\}$$

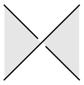
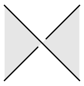
s.t. regions sharing an edge are always mapped to different colors. (A region is called a connected component of the complement of the plane curve of D.)

The *checkerboard graph* $G(D)$ of D is defined to have vertices corresponding to black regions in the checkerboard coloring of D, and an edge for each crossing p of D connecting the two black regions opposite at crossing p (so multiple edges between two vertices are allowed).

The construction of the checkerboard graph defines a bijection

$$\{\text{ alternating diagrams up to mirroring }\} \quad \longleftrightarrow \quad \{\text{ plane graphs up to duality }\}.$$

We call the inverse of G, reconstructing D from $G(D)$, by D, i.e. $D(G(D)) = D$.

Duality of the plane graph corresponds to switching colors in the checkerboard coloring and has the effect of mirroring the alternating diagram if we fix the sign of the crossings so that each crossing looks like  rather than .

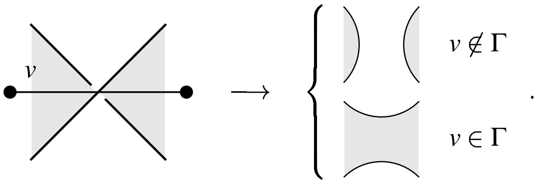
Remark 4 D has no nugatory crossings $\iff$ $G(D)$ has no loop edges and is 2-connected (i.e., the removal of any single edge does not disconnect it).

The importance of the determinant in our graph theoretical context lies in the following

Lemma 3 $\det(D)$ is the number of spanning trees in a checkerboard graph of D for any alternating link diagram D.

Proof. As mentioned, by the Kauffman bracket definition of the Jones polynomial V, for an alternating diagram D, the determinant $\det(D) = |\Delta_D(-1)| = |V_D(-1)|$ can be calculated by counting the monocyclic states of D.

Let Γ be a spanning tree of the checkerboard graph G of D. Define a state $S(\Gamma)$ as follows: for any edge v in G set



(9)

Then it is easy to check that S gives a bijection between monocyclic states of D and spanning trees of G. $\square$

4.3. The Conway polynomial

There is a way to fix the multiplicative ambiguity in the definition of Δ_D by a specific choice of normalization. This naturally happens when introducing Δ as a reparametrization of Conway's polynomial [9] $\nabla \in \mathbb{Z}[z]$. ∇ can be defined by $\nabla(\bigcirc) = 1$ and the skein relation $\nabla_+ - \nabla_- = z\nabla_0$. Then set

$$\tilde{\Delta}(t) := \nabla(t^{1/2} - t^{-1/2}) \in \mathbb{Z}[t^{\pm 1/2}]. \tag{10}$$

$\tilde{\Delta}$ turns out to be a specific normalization of Δ, i.e. $\tilde{\Delta}_D \doteq \Delta_D$. See Theorem 6.9 of [17]. $\tilde{\Delta}$ is usually still denoted by Δ, but it is helpful to distinguish here between the normalized and unnormalized version.

It follows from the skein relations that $[\nabla_L(z)]_{z^i} = 0$ if i has the same parity as the number $c(L)$ of components of L, and that $z^{c(L)-1} \mid \nabla_L(z)$ for a link L. This fact has several consequences, summarized here, since they will be used in the sequel.

First, wee see that

$$\tilde{\Delta}(1/t) = (-1)^{c(L)-1}\tilde{\Delta}(t). \tag{11}$$

Also $\operatorname{span}\tilde{\Delta}_L = \max\deg\nabla_L$, and it has the opposite parity to $c(L)$.

We also have that $(t^{1/2} - t^{-1/2})^{c(L)-1} \mid \tilde{\Delta}_L(t)$, and in particular

$$2^{c(L)-1} \;\Big|\; |\Delta_L(-1)|; \tag{12}$$

The converse is not true in general, that is, $2^{1-c(L)}|\Delta_L(-1)|$ may be even (and non-zero).

However, this does not happen if L is a knot. For a knot K we have the special form

$$\nabla_K(z) \in 1 + z^2\mathbb{Z}[z^2].$$

Thus

$$\tilde{\Delta}_K(1) = \nabla_K(0) = 1. \tag{13}$$

Also

$$\tilde{\Delta}_K(-1) = \nabla_K(2i) \equiv 1\,(4) \qquad (i = \sqrt{-1}). \tag{14}$$

Since $\det(K) = |\Delta_K(-1)|$, it follows from (13) that a link has odd determinant if (and by (12) only if) it is a knot.

Finally, note also that (10) can be used to define ∇ from $\tilde{\Delta}$.

5. Unifying Δ_G and Δ_D

Definition 5 An oriented link diagram is called *positive/negative*, if all its crossings look like  or all look like . A link is called positive/negative if it has a positive/negative diagram.

Definition 6 A region of a diagram is called *Seifert circle region* (resp. non-Seifert circle, or *hole* region), if any two neighbored edges in ∂R (i.e., such sharing a crossing) are equally (resp. oppositely) oriented (between clockwise or counterclockwise) as seen from inside R. A diagram is called *special* iff all its regions are (either) Seifert circle regions or hole regions.

It is a well-known fact (see e.g. [10, §1] or Lemma 9.2, p. 128 of [17]) that two of the three properties of a link diagram to be positive/negative, alternating and special imply the third. Thus in an alternating positive/negative diagram D each region is either a Seifert circle region or a hole region, and (hence) among the two regions bounded by an edge exactly one is of either types. Then the partition of the regions of D into Seifert circle regions and hole regions coincides with the partition into black and white regions in a checkerboard coloring of D.

The importance of special alternating diagrams to our context comes from

Proposition 4 Let G be a plane connected graph. Then

G is even valence	$\Longleftrightarrow$	$D(G)$ can be oriented to be special, and with this orientation G is this one of the two checkerboard graphs of $D(G)$, whose vertices correspond to the non-Seifert circle regions of $D(G)$.

Proof. '$\Longleftarrow$' Since any two neighbored edges bounding a hole region R in a special alternating diagram D are oppositely oriented w.r.t. R, the number of such edges must be even for all R. Then G is even valence.

'$\Longrightarrow$' Contrarily, let G be even valence (equivalently, G^* be bipartite) and consider $D(G) = D(G^*)$. Let the vertices of G correspond to w.l.o.g. black regions in a checkerboard coloring of $D(G)$. I.e., the checkerboard graph with vertices in the white regions of $D(G)$ is G^*, which is bipartite.

Fix a white region R. Orient all edges in its boundary clockwise. Then proceed by induction as follows: whenever the edges of a white region S are oriented (as seen from inside S) clockwise (resp. counterclockwise), orient the edges of the boundary of a white region T opposite to S at some crossing counterclockwise (resp. clockwise). Because G^* is bipartite, this ensures that the choice of orientation of ∂T is independent on the way we arrive to orient ∂T.

Since every edge in $D(G)$ bounds exactly one white region, there is no ambiguity of assigning an orientation to a given edge from the two regions it bounds, and each edge is oriented. A local picture at each crossing then shows that this edge orientation defines an orientation of the link. □

We are now prepared to show the result which makes it possible to link knot and graph theory.

Theorem 2 Let G be an even valence plane connected graph, and $D = D(G)$ its associated special alternating link diagram by proposition 4. Then

$$\Delta_G(t) \doteq \Delta_D(-t)\,.$$

Proof. Fix two adjacent regions $R'_{1,2}$ in D. Let R'_2 be the hole region among both. We have then

$$\Delta_D(t) \doteq \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^{n} A_{i,\sigma(i)}\,, \tag{15}$$

with the sum taken *de facto* over all $\sigma \in S_n$ s.t. $c_{\sigma(i)} \in \partial R_i$ for all $i = 1,\dots,n$. We call such σ *good*.

As discussed in §4.1, the number of such σ is $\det(D) = \varepsilon\Delta_D(-1)$ for some $\varepsilon = \pm 1$. Then it follows from (15), that for all σ the summands

$$(-1)^\sigma \prod_{i=1}^{n} A_{i,\sigma(i)}$$

are of the form $\varepsilon\cdot(-t)^{n_\sigma}$ for certain numbers n_σ (otherwise cancellations will occur).

Order the $R_1,\dots,R_n$ so that $R_1,\dots,R_s$ are the Seifert circle regions. Then the r.h.s. of (15) can be split as

$$\sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=s+1}^{n} A_{i,\sigma(i)} \cdot \prod_{i=1}^{s} A_{i,\sigma(i)}\,.$$

But because $R_1,\dots,R_s$ are Seifert circle regions, for any $1 \le i \le s$, $A_{i,p}$ is independent on p as long as $p \in \partial R_i$. Consequently, the last factor is (a unit and) independent on σ. (This holds in greater generality; see Theorem 6.7 of [17].) Thus

$$\Delta_D(t) \doteq \sum_{\sigma\in S_n \text{ good}} (-1)^\sigma \prod_{i=s+1}^{n} A_{i,\sigma(i)}\,.$$

Still the $\det(D)$ summands are non-zero, and are of the form $\varepsilon'\cdot(-t)^{n'_\sigma}$.

What remains to show now is that under the bijection

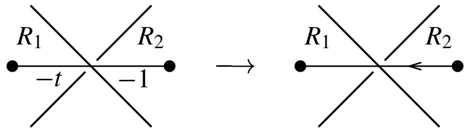
$$\sigma \quad\longleftrightarrow\quad \Gamma = \Gamma(\sigma)$$

between an arrow assignment σ in D (giving a non-zero summand) and a spanning tree $\Gamma \subset G(D)$, given by composing (8) and (9), we have

$$\prod_{i=s+1}^{n} A_{i,\sigma(i)} \quad = \quad \pm t^{i(\Gamma,G,v)}$$

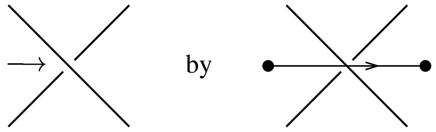
with v in G corresponding to R_2' in D (note that the vertices of G correspond to the non-Seifert circle regions in D).

Now two neighbored corners (corners or neighbored crossings) of a non-Seifert circle region in D receive markings ± 1 and $\pm t$ in (5). Orient an edge in G corresponding to a crossing with markings ± 1 and $\pm t$ from the region with marking ± 1 to the region with marking $\pm t$:



Then we obtain an alternating edge orientation of G.

Similarly, orient every edge in $\Gamma(\sigma)$ corresponding to an arrow



(16)

Then this defines the canonical orientation of Γ. Thus the number of factors $\pm t$ in

$$\prod_{i=s+1}^{n} A_{i,\sigma(i)}$$

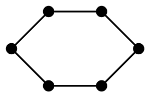
is exactly $i(\Gamma, G, v)$, and the theorem is proved. $\square$

Remark 5 The theorem, in the way it is proved, holds with Δ_G meant for one specific choice of root $v \in V(G)$ and alternating edge orientation of G. However, we already proved in proposition 1 (and shall reprove it in different terms shortly), that Δ_G does not depend on the choice of root. Because of remark 2 the same is true for the choice of alternating edge orientation. We will soon see also that Δ_G is independent on the choice of planar embedding of a given planar even valence graph G, so that it can be defined for a *planar* (and not just plane) graph.

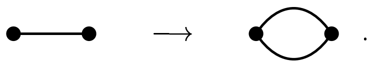
6. Properties of the polynomial of a planar graph

In the special case we consider, knot theory can say much more on Δ_G than what we proved in §2 for the general case. First, we need some more definitions.

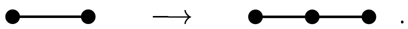
Definition 7 A *chain* is a connected graph with all vertices of valence 2.



The *doubling* of an edge is the operation



A *bisection* of an edge is the operation



(Other edges may be incident from the left- and rightmost vertices on both sides, and the rest of the graph is assumed to be equal.)

A graph is *series-parallel* (or abbreviatedly *SP*) if it can be obtained from by iterated doubling and bisecting of edges.

The following theorem now transcribes what we know, or can derive, knot-theoretically in graph-theoretic terms. Beside other values of the graph polynomial, we obtain several inequalities and congruence relations for the number of spanning trees.

Theorem 3 Let G be a connected plane even valence graph with an alternating edge orientation, and $\Delta = \Delta_G$ its polynomial. Then the following holds.

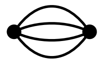
1) $\Delta_G(t)$ is independent on the choice of root v.

2) $\min\deg\Delta = 0$, $\max\deg\Delta = v(G) - 1$.

3) $\Delta(t^{-1}) = \Delta(t) \cdot t^{1-v(G)}$, or equivalently, Δ_G is independent on which of either alternating edge orientations of G is chosen.

4) Δ_G is independent on the planar embedding of the planar graph of G.

5) $\max\mathrm{cf}\,\Delta = 1 \iff G$ is a join of chains.
$\max\mathrm{cf}\,\Delta = 2 \iff G$ is a join of chains and exactly one copy of the graph

$M =$  ,

or a graph obtained from M by bisecting edges.

6) Define
$$\tilde{\Delta}_G(t) := (-\sqrt{t})^{-\max\deg\Delta_G}\Delta_G(-t) \in \mathbb{Z}[t^{\pm 1/2}]. \tag{17}$$
Then for $\Delta_G(1)$ odd
$$\frac{e(G)^2}{8} \geq \frac{1}{2}\tilde{\Delta}''(1) \geq \max\deg\Delta, \tag{18}$$
and
$$\frac{1}{2}\tilde{\Delta}''(1) \geq \frac{e(G)}{4} \tag{19}$$
if G has (additionally) no loop edges and is 2-connected.

7)
$$\Delta(-1) = \begin{cases} 0 & \Delta(1) \equiv 0\,(2) \\ 1 & \Delta(1) \equiv 1\,(2) \end{cases}$$

8) Let $\Delta(1)$ be odd. Then $4 \mid (\Delta(1) - v(G))$, or equivalently, $v(G)$ is also odd, and
$$\begin{aligned} \Delta(1) \equiv 3\,(4) &\iff v(G) \equiv 3\,(4) \\ \Delta(1) \equiv 1\,(4) &\iff v(G) \equiv 1\,(4)\,. \end{aligned}$$

If $\Delta(1) \equiv 2\,(4)$, then $v(G)$ is even.

9) Define a polynomial $\nabla_G(z) \in \mathbb{Z}[z]$ by
$$\nabla_G(t - t^{-1}) := \tilde{\Delta}_G(t^2) = (-t)^{-\max\deg\Delta}\Delta_G(-t^2)\,. \tag{20}$$
Then ∇ has coefficients in z^i with $i \leq \max\deg\nabla = \max\deg\Delta$ and $2 \mid \max\deg\nabla - i$.

   a) ∇ is positive, i.e. for each $i \in [\min\deg\nabla, \max\deg\nabla]$ with $2 \mid \max\deg\nabla - i$, $[\nabla(z)]_{z^i} > 0$.

   b) For any $i > 0$ there are (explicitly computable) constants C_i such that for any plane even valence graph G it holds $[\nabla_G]_{z^i} \leq C_i\, e(G)^i$. Moreover, if $\Delta(1)$ is odd, we can set (for k even) $C_i = {}^1/\sqrt{8}^k$

10) a) $\Delta(1) \le \delta_0^{e(G)}$ with $1/\delta_0 \approx 0.543689$ being the real positive zero of $f(x) = x^3 + x^2 + x - 1$.

  b) If G is additionally series-parallel, then $\Delta(1) \le F_{e(G)}$ with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ being the Fibonacci numbers. In particular,

$$\Delta(1) \le \left(\frac{1+\sqrt{5}}{2}\right)^{e(G)}.$$

11) If $e(G) > 1$, then $0 < [\Delta]_{t^0} \le [\Delta]_{t^1}$.

12) For any $\min\deg \nabla_G \le i \le \max\deg \nabla_G$ with $\max\deg \nabla_G - i$ even, we have $[\nabla_G]_{i-2}[\nabla_G]_{i+2} < [\nabla_G]_i^2$.

13) The coefficients $[\nabla_G]_i$ of ∇_G are algebraically independent for $i > 0$ of the opposite parity to $v(G)$ (when considered on all graphs G of fixed parity of $v(G)$). Similarly, except for the identity $[\tilde\Delta_G]_{-i} = (-1)^{2i}[\tilde\Delta_G]_i$, the same holds for the coefficients of $\tilde\Delta_G$ for $2i$ of the opposite parity to $v(G)$.

14) If G is additionally self-dual (i.e. coincides with G^* up to changes of the unbounded cell), then

$$\Delta_G(1) \ge \frac{1}{4} e(G)\left(e(G) - 6\right),$$

and $\Delta_G(1)$ is even (unless $v(G) = 1$).

Proof. Although some of the points have been proved in §2 before, we give slightly different alternative knot-theoretical arguments for them.

1) Varying v corresponds to varying R_2' in the proof of theorem 2. But by Alexander's work Δ_D depends on the choice of R_2' only up to a unit. Hence by theorem 2, Δ_G depends on the choice of R_2' only up to a unit as well. That this unit must be $+t^n$, and not $-t^n$, is clear since all coefficients of Δ_G are positive. Since clearly for any choice of root v, $\min\deg \Delta_G \ge 0$ and $\max\deg \Delta_G \le v(G) - 1$, the rest follows from the next point in the proof (if the unit were t^n with $n \ne 0$, then $\operatorname{span} \Delta_G \le v(G) - 1 - |n|$).

2) It suffices to show that $\operatorname{span} \Delta = v(G) - 1$. Let $s(D)$ be the number of Seifert circle(region)s of D and $c(D) = e(G)$ its crossing number. Then D has $r(D) = c(D) + 2$ regions, and

$$v(G) - 1 \,=\, r(D) - s(D) - 1 \,=\, c(D) - s(D) + 1 \,=\, 1 - \chi(D),$$

where $\chi(D)$ is the Euler characteristic of the Seifert surface obtained by applying the Seifert algorithm to D. It is a known fact [11, 24], that for an alternating diagram D, $\operatorname{span} \Delta(D) = \max\deg \nabla_D = 1 - \chi(D)$.

3) (11) implies that $[\Delta_G]_{t^k} = \pm[\Delta_G]_{t^{\max\deg \Delta - k}}$ for any k. That the sign must be always positive is clear since both coefficients are non-negative. Thus, in particular we have proved now that Δ_G is independent on both the choice of root and alternating edge orientation of G.

4) By the work of Whitney [41, 42] (see also [4, proof of proposition 1.2] and [21, corollary 6]), there is a set of moves transforming a planar embedding of a given planar graph G into another one [21, fig. 7]. As remarked on p. 107 bottom–109 *ibid.*, all these moves have the effect of mirroring or applying a mutation to $D(G)$, which does not alter the Alexander polynomial (orientation reversal and change of the way to build a connected sum are also special types of mutation).

5) This claim states that a fibered special alternating diagram D is the connected sum of $(2,n)$-torus link diagrams. For this see e.g. proposition 13.25 in [6], or [37, 10]. The second claim states that if the leading coefficient of $\Delta(D)$ is ± 2, then D's prime factor decomposition has exactly one copy of a (p,q,r,s)-pretzel diagram (with the twists in the groups of p,q,r,s parallel). For this see lemma 4.3 of [26]. An alternative proof was given in [34].

6) We have from (17), Theorem 2 and the discussion in §4.3 that

$$\tilde\Delta_G(t) \doteq \Delta_G(-t) \doteq \Delta_D(t) \doteq \tilde\Delta_D(t)\,.$$

Moreover, from (17) and parts 2 and 3 of the theorem (already proved) it follows that

$$\tilde\Delta_G(t) = \pm\tilde\Delta_G(1/t)\,.$$

Similarly (11) gives the same for $\tilde\Delta_D$, so that $\tilde\Delta_D = \pm\tilde\Delta_G$. From (17) we have $\max\mathrm{cf}\,\tilde\Delta_G > 0$, and similarly $\max\mathrm{cf}\,\tilde\Delta_D = \max\mathrm{cf}\,\nabla_D > 0$ follows from [10, corollary 2.1] (see also the proof of point 9). Thus $\tilde\Delta_D(t) = \tilde\Delta_G(t)$.

Now, $1/2\tilde\Delta''(1)$ is the Casson invariant (or Vassiliev invariant of degree 2). The inequalities (18) and (19) are consequences of the Gauß sum formulas for knots: the first inequality of (18) follows from theorem 1.E of [29], and the second inequality of (18) and the inequality (19) follow from [32] (see also remark 4).

7) If $\Delta(1) \equiv 0(2)$, then D is a link diagram (of > 1 component), and hence $\Delta_G(-1) = \pm\Delta_D(1) = 0$ (see §4.3). If $\Delta(1) \equiv 1(2)$, then D is a knot diagram. Then

$$\Delta_G(-1) = (-1)^{\max\deg\Delta_G}\tilde\Delta_G(1) = \tilde\Delta_G(1) = \tilde\Delta_D(1) = 1\,,$$

since $\max\deg\Delta_G$ is even for a knot diagram D.

8) If $\Delta_G(1)$ is odd, then D is a knot. We have

$$\Delta_G(1) = \det(D) = |\Delta_D(-1)|\,.$$

Because of (14) and [25] we have for the signature $\sigma(D)$ the equivalences

$$\begin{array}{ccccc} \tilde\Delta_D(-1) < 0 & \iff & \det(D) \equiv 3\,(4) & \iff & \sigma(D) \equiv 2\,(4) \\ \tilde\Delta_D(-1) > 0 & \iff & \det(D) \equiv 1\,(4) & \iff & \sigma(D) \equiv 0\,(4)\,. \end{array} \tag{21}$$

But by [25] for a special alternating diagram D,

$$\sigma(D) = \operatorname{span}\Delta(D)\,, \tag{22}$$

and we have by the result proved in part 2

$$\operatorname{span}\Delta(D) = v(G) - 1\,. \tag{23}$$

Putting together (21), (22) and (23) proves the first assertion.

If $\Delta_G(1) \equiv 2\,(4)$, then by §4.3 and (12) D is a two component link diagram, and $\operatorname{span}\Delta_D$ is odd. Thus by (23), $v(G)$ is even.

9) By the said in the beginning of part 6, (10) and (20), we have $\nabla_G(z) = \nabla_D(z)$.

Claim a) follows from the result in [35] (proved, however, to a large extent already in [10, corollary 2.1]), that all coefficients between $z^{c(L)-1}$ and $z^{\max\deg\nabla_L}$ of ∇ of a (non-split) positive link L of $c(L)$ components are strictly positive. This was proved there only for $c(L) \le 2$, but to show it for all higher $c(L)$, one uses induction on $c(L)$ and applies the skein relation for ∇ on mixed crossings of a positive diagram of L (i.e., crossings involving two different components of L).

Claim b) follows because $[\nabla]_{z^i}$ is a Vassiliev invariant of degree i [2], and the proof of the Lin-Wang conjecture [20] for links given in [35] (and previously for knots in [3], which does not suffice here, though). Then for any n there is a constant $C_{i,n}$ such that

$$[\nabla_D]_{z^i} \le C_{i,n}\,c(D)^i$$

for any diagram D of n components and $c(D) = e(G)$ crossings. (It follows from the proof of [35] that these constants are computable for any given i and n, although the computation may be very difficult.) Since by the said in §4.3, $[\nabla_D]_{z^i} = 0$ for $n > i+1$, $C_i := \max\limits_{n\le i+1} C_{i,n}$ gives the desired constants. For the second assertion, use the first inequality in (18) and the inequality in part 12.

10) This follows from Theorem 2.1 of [36] (G is series-parallel $\iff$ $D(G)$ is arborescent).

11) The first inequality is clear. The second one was proved more generally in proposition 2.

12) This will be proved in [38].

13) This proof uses the concept of braiding sequences of [39]. (Such an argument has been applied extensively, and can be found in more detail there.)

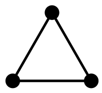
Consider first $v(G)$ odd. We show that the independence holds already if we restrict ourselves to graphs G giving knot diagrams (i.e. with $\Delta_G(1)$ odd). Let $P \in \mathbb{Q}[x_1,\dots,x_l]$ be some non-trivial polynomial, and $v_i = [\nabla]_{n_i}$ for some even numbers $2 \le n_1 < \dots < n_k$. Since, as well-known, any polynomial in $1+z^2\mathbb{Z}[z^2]$ is the Conway polynomial of a knot K, take some K with $P(K) = P(v_1,\dots,v_k)(K) \ne 0$. As shown in [39], K has a special diagram D. (The result may be found also in [6].) Consider the braiding sequence associated to D with antiparallel braidings at each crossing. Then $P(D(x_1,\dots,x_l))$ is a polynomial in $x_1,\dots,x_l$, which does not vanish. (Here $x_1,\dots,x_l$ are odd integers, and l is the crossing number of D.) Then there exist $x_1,\dots,x_l$ all positive for which $P(D(x_1,\dots,x_l)) \ne 0$, and hence $D(x_1,\dots,x_l)$ is a special alternating diagram, on which $P(v_1,\dots,v_k) \ne 0$.

For $\tilde\Delta$ one argues in the same way. We have that for any polynomial $\tilde\Delta$ with $\tilde\Delta(t) = \tilde\Delta(1/t)$ and $\tilde\Delta(1) = 1$ there is a knot K with $\Delta_K = \tilde\Delta$. Take a special diagram D of an appropriate K. By choosing the braidings to be antiparallel, one bounds $\max\deg\Delta$ on all diagrams in the braiding sequence associated to D, and hence each coefficient of $\Delta(D(x_1,\dots,x_l))$ is a polynomial in $x_1,\dots,x_l$.

To cover the case of $v(G)$ even, one needs to consider even number of components (when the powers in ∇_G are odd, and those in $\tilde\Delta_G$ are fractional). By the result of Kondo [18] any polynomial in $1+z^2\mathbb{Z}[z^2]$ is the Conway polynomial of an unknotting number one knot K. Thus any polynomial in $z\mathbb{Z}[z^2]$ is the Conway polynomial of a two-component link. Then the same argument applies.

14) If $G = G^*$, then $e(G)$ is even and $D(G)$ depicts an achiral link. Then apply [33, Proposition 5.3]. The achirality condition following from the self-duality of G (and the one needed to apply the result of [33]) is the most general one, allowing the isotopy taking the link to its mirror image to interchange and/or preserve or reverse the orientation of the components in an arbitrary way (we call this here 'weak achirality'). The chirality results for positive links of (or following from) [8, 10, 31, 37, 43, 45] address just the restricted notion of achirality where the isotopy is required to preserve or reverse the orientation of all components *simultaneously*. If, however, $\Delta(1) \equiv 1\,(2)$ (and $v(G) > 1$), then D is a (non-trivial) knot diagram, for which both notions of achirality coincide, and thus the *op. cit.* results give a contradiction. $\square$

Example 1

G = 

$$\Delta_G = 1+t+t^2\,,\quad \tilde\Delta_G = t^{-1}-1+t\,,\quad \nabla_G = 1+z^2\,.$$

Corollary 1

$$[\Delta]_{t^k} \le \tilde C_{\max\deg\Delta}\, e(G)^{\max\deg\Delta} \binom{\max\deg\Delta}{k}\,,$$

for some constants $\tilde C_i > 0$, and hence

$$\Delta(1) \le \tilde C_{\max\deg\Delta}\,(2e(G))^{\max\deg\Delta}\,.$$

Proof. By part 9 of the above theorem, we have

$$\begin{aligned}
\pm[\Delta]_{t^k} &= [\Delta(-t^2)\,t^{-\max\deg\Delta}]_{t^{2k-\max\deg\Delta}} \\
&= [\nabla(t-t^{-1})]_{t^{2k-\max\deg\Delta}} \\
&= \sum_i [\nabla]_i \left[(t-t^{-1})^i\right]_{2k-\max\deg\Delta}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i\leq v(G)-1} C_i\, e(G)^i \binom{i}{\frac{i-2k+\max\deg\Delta}{2}} \\
&= \sum_{\substack{\max\deg\Delta - 2k \leq i \leq \max\deg\Delta \\ 2 \mid \max\deg\Delta - i}} C_i\, e(G)^i \binom{i}{\frac{i-2k+\max\deg\Delta}{2}} \\
&\leq \tilde{C}_{\max\deg\Delta} e(G)^{\max\deg\Delta} \cdot \binom{\max\deg\Delta}{k},
\end{aligned}$$

with

$$\tilde{C}_d := 2\max_{d'\leq d} C_{d'}\,.$$

For the last estimate use that

$$b(i,k) := \binom{i}{\frac{i-2k+\max\deg\Delta}{2}}$$

satisfies $b(i+2,k)\geq 2b(i,k)$, and that

$$1+\frac{1}{2e(G)}+\frac{1}{(2e(G))^2}+\ldots = \frac{2e(G)}{2e(G)-1} \leq 2$$

for $e(G)\geq 1$ (for $e(G)=0$ the claim is trivial). □

7. Open problems and conjectures

We conclude with some conjectures and questions. First consider the case of plane graphs G with alternating edge orientation.

7.1. The trapezoidal conjecture

The first conjecture points to a possible generalization of part 11 of theorem 3.

Conjecture 1 Δ_G is trapezoidal. Here a polynomial $f(t)=\sum_{i=0}^m c_i t^i \in \mathbb{Z}[t]$ is said to be trapezoidal if $t^m f(1/t) = f(t)$, $0 < c_0 \leq c_1 \leq \cdots \leq c_{[m/2]}$, and whenever $c_i = c_{i+1}$ for some $i < [m/2]$, then $c_i = c_j$ for any $i \leq j \leq [m/2]$.

This is a special case of a long-standing problem of Fox [12] on the Alexander polynomial of an alternating link. Except for the first author's results [23] and the preceding work of Hartley [14], there is little progress towards this conjecture. An attempt to prove conjecture 1 (or equivalently, Fox's conjecture for special alternating links) is in fact what led to the investigations described in this paper.

7.2. Maximal number of spanning trees

Conjecture 2

$$\sup_G \sqrt[e(G)]{\Delta_G(1)} = \delta_0 \approx 1.8393\,.$$

Here the supremum is taken over all connected plane/planar even valence graphs G, and δ_0 is the number occurring in part 10a) of theorem 3.

This is, in a stronger form, a conjecture of the second author on the growth of the determinant of links of given crossing number. From [36], the inequality

$$\sqrt[e(G)]{s(G)} \leq \delta_0$$

follows for arbitrary planar graphs G ($s(G)$ denoting the number of spanning trees of G), and it was conjectured that this inequality is sharp. Our conjecture 2 here is stronger, since we restrict ourselves just to even valence (or equivalently bipartite) planar graphs. However, the examples considered in [36, §4] as candidates to attain this supremum, and giving the estimate

$$\sup_G \sqrt[e(G)]{\Delta_G(1)} \geq 1.7897 > \delta_0 - 0.05\,,$$

are of this type.

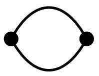
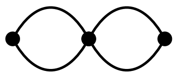
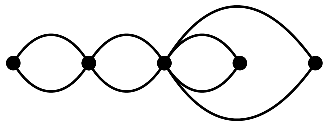
Question 1 Is

$$\sup_{G \text{ SP}} \sqrt[e(G)]{\Delta_G(1)} = \frac{1+\sqrt{5}}{2} \ ? \tag{24}$$

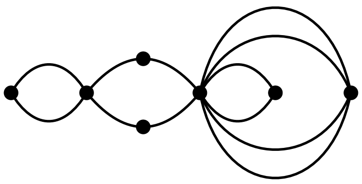
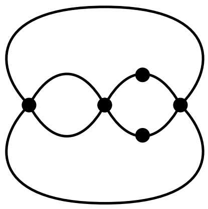
It was also proved in [36] that, replacing '$\Delta_G(1)$' by '$s(G)$' and taking the supremum over all planar graphs G, equality in (24) holds. The graphs of [36], realizing the supremum, however, are not even valence. This motivates the above question.

7.3. Weakly achiral positive alternating links

Contrarily to the said in the proof of part 14 of Theorem 2, there are weakly achiral positive alternating links, connected sums of Hopf links being examples. Some of their diagrams give rise to plane even valence self-dual graphs, e.g.

 ,  , and  .

There are other examples, though:

 and  .

Question 2 Is it possible to give a description of all positive weakly achiral alternating links and all self-dual even valence plane graphs (which can be derived from them)?

7.4. Further partial generalizations of theorem 3

Another problem is that some of the results knot-theoretically proved in theorem 3 only for the planar case may extend to arbitrary even valence graphs.

Now consider an arbitrary even valence graph G, not necessarily planar, with fixed canonical edge orientation. It turns out that (beside the parts of theorem 3 proved in this more general case), for all examples so far considered:

- At least 8), 9a) and 10a) of theorem 3 and conjecture 1 always hold.

- In 6) of theorem 3, the second inequality in (18), and (19) hold (even for arbitrary $\Delta(1)$), but not the first inequality in (18) (examples are the 4-valent graphs of the alternating diagrams of the knots 9_{27}, 9_{29}, 9_{30} in [30, appendix] with edge orientation given by knot orientation).

- Part 7 of theorem 3 holds with the modification that $\Delta(-1)$ may be > 1. In general it can be any square. (Let G be a chain of length 3, with cyclic edge orientation, and replace each edge in G by n parallel edges, all of the same orientation as the original one.) Can it be anything else?

There is, however, no explanation so far why these properties of theorem 3 suggested to extend in this more general case do so.

Question 3 Does any of the above mentioned properties in theorem 3 generalize to the case of arbitrary even valence graphs?

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