THE JONES POLYNOMIAL, GENUS AND WEAK GENUS OF A KNOT

This is a preprint. I would be grateful for any comments and corrections!

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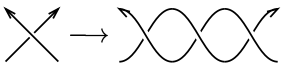
1. Introduction

In his book [Ad, p. 105 bottom], C. Adams mentions a result of Morton that there exist knots, whose genus g is strictly less than their weak genus $\tilde{g}$, the minimal genus of (the surface of Seifert's algorithm applied on) all their diagrams. This observation appears just as a remark in [Mo], but was very striking to the author. Motivated by Morton's example, the author started in a series of papers [St2, St, St3] the study of the invariant $\tilde{g}$. A key role in what we can say so far about $\tilde{g}$ plays [St2, theorem 3.1], saying that knots of given $\tilde{g}$ decompose into finitely many sequences of the kind introduced in [St4], and called there "braiding sequences", that is, can be obtained from finitely many diagrams by successive applications of antiparallel twists at a crossing


. (1)

This theorem has several direct consequences, *inter alia*, to the enumeration of such knots or the properties of their knot polynomials.

In this paper, we extend the series of boundedness and stability criteria for the Jones polynomial V [J], presented in [St] for positive knots, to alternating knots. We make more precise our observation of [St3], that any coefficient of V of an alternating knot has an upper bound, which is polynomial in the crossing number for fixed genus, by writing down an explicite estimate. Furthermore, we show that the value range of any sequence of fixed length of leading or trailing coefficients of V of an alternating knot of given genus stabilizes as its crossing number goes to infinity.

Both properties are generalized in slightly weaker forms to non-alternating knots. Finally, we use these extensions to generalize Morton's example to a series of knots with fixed genus, but arbitrarily high weak genus. Thus, unfortunately, no control from below can be expected on g from $\tilde{g}$.

It is to be expected that a proof for specific series of examples is possible by skein module calculations also using Morton's inequality [Mo] $\max\deg_v P/2 \le \tilde{g}$ involving the maximal degree of the v variable in the HOMFLY polynomial P [H]. We decide here, however, to present a criterion using the Jones polynomial (and more exactly the Kauffman bracket), whose derivation is more analytical.

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2. Preliminaries

The Jones polynomial [J] is a Laurent polynomial in one variable t (more precisely in its square root) associated to an oriented knot or link in S^3 and can be defined by being 1 on the unknot and the (skein) relation

$$t^{-1}V_+ - tV_- + (t^{-1/2} - t^{1/2})\,V_{)(} \,=\, 0\,, \tag{2}$$

with $V_+, V_-, V_{)(}$ denoting diagrams equal except near one crossing, which is resp. positive, negative and smoothed out.

Briefly after Jones's discovery, Kauffman [Ka] found another definition of this invariant called "Kauffman's state model" or "Kauffman bracket" (see also [Ad, §6.2]).

Recall, that the Kuaffman bracket $\langle D\rangle$ of a diagram D is a Laurent polynomial in a variable A, obtained by summing over all states the terms

$$A^{\#A-\#B}\left(-A^2-A^{-2}\right)^{|S|-1}, \tag{3}$$

where a state is a choice of splittings of type A or B for any single crossing (see figure 1), $\#A$ and $\#B$ denote the number of type A (resp. type B) splittings and $|S|$ the number of (disjoint) circles obtained after all splittings in a state.

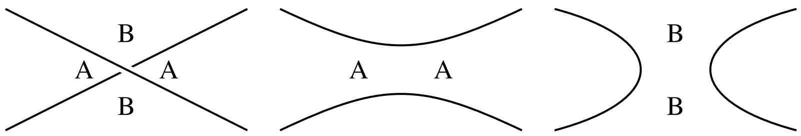
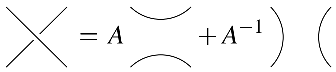


Figure 1: The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing.

The Jones polynomial of a link L is related to the Kauffman bracket of some diagram of it D by

$$V_L(t) \,=\, \left(-t^{-3/4}\right)^{-w(D)}\,\langle D\rangle\Big|_{A=t^{-1/4}}\,. \tag{4}$$

The Kauffman bracket skein module of a room (a disc with a distinguished number of points on its boundary) is the module, say, over $\mathbb{Z}$, generated by isotopy classes of inhabitants of this room (tangle diagrams in this disc, intersecting its boundary exactly in the distinguished points), and with relations corresponding to resolving the crossings according to the Kauffman bracket relation.



See, e. g., [BFK].

The concept of a braiding sequence was introduced in [St4] in the context of Vassiliev invariants, but subsequently turned out to be more useful in a special case when considering knot diagrams, on which the Seifert algorithm [Ad, §4.3] gives a surface of given genus. (We subsequently call this genus the genus of the diagram.)

Definition 2.1 A $\bar{t}_2$-move is the move in a diagram D is a replacement of (a neighborhood of) some distinguished crossing in D by the tangle of 3 antiparallely twisted crossings, as shown in (1).

A braiding sequence associated to a diagram is a family of diagrams, parametrized by $c(D)$ odd numbers $x_1,\dots,x_{c(D)}$ (where $c(D)$ henceforth denotes the number of crossings of D), each one indicating the number of $\bar{t}_2$ moves performed at each crossing. We adopt the convention that for $x_i<0$ we switch the crossing numbered by i and apply $(-x_i-1)\,\bar{t}_2$ moves on the switched crossing.

We consider crossings as equivalent, if they form a revsere clusp, so that $\bar{t}_2$ on either of them have the same effect on the diagram. The maximal number of (such equivalence classes of) crossings over diagrams of genus g we call d_g.

Theorem 2.1 (theorem 3.1 of [St2]) Knot diagrams of given genus decompose into finitely many equivalence classes modulo $\bar{t}_2$ moves and their inverses. That is, they all can be obtained from finitely many (called "generating") diagrams by repeated $\bar{t}_2$ moves.

3. The Jones polynomial of alternating knots of given genus

Directly from [St2, theorem 3.1], in the proof of theorem 9.3 of [St] we mentioned a way how to compute V on a whole braiding sequence from the Jones polynomials of the generating diagram (as defined in [St2]) and all its crossing-changed versions. From this principle, the following observation is relatively straightforward, but in view of the results of [St3, §6] maybe should be recorded in its own right.

Theorem 3.1 There exists a constant C, such that for any alternating knot K and any $k \in \mathbb{Z}$ it holds

$$\left|[V_K(t)]_{t^k}\right| \leq \max_{2g(K)+1 \leq k \leq d_{g(K)}} \left[\frac{Cc(K)}{k}\right]^k, \tag{5}$$

where $c(K)$ denotes the crossing number of K and $g(K)$ its genus, $[V]_{t^k}$ is the coefficient of t^k in V, and $d_{g(K)}$ can be defined by

$$d_{\tilde{g}} := \min\left\{ i \in \mathbb{N} : \limsup_{n\to\infty} \frac{\left|\mathcal{A}_{n,\tilde{g}}\right|}{n^i} = 0 \right\}, \tag{6}$$

with

$$\mathcal{A}_{n,\tilde{g}} := \{\, K \text{ alternating}, g(K) = \tilde{g}, c(K) = n \,\}. \tag{7}$$

Remark 3.1 For fixed $c(K)$ the maximal value on the right of (5) is attained at $k = c(K)/e$, which is exponential in $c(K)$. Therefore, the essence of this theorem is the claim that the coefficients of V for K alternating grow polynomially in $c(K)$ for fixed $g(K)$. This was already noted in [St3], but here we give this more explicite estimate.

Proof. This is basically a repetition of the proof of theorem 9.3 in [St]. If V_n denote the Jones polynomials of L_n, where L_n are links with diagrams D_n equal except in one room, where n antiparallel half-twist crossings are inserted, then from the skein relation for the Jones polynomial we have

$$V_{2n+1}(t) = t^{2n}V_1(t) + \frac{t^{2n}-1}{t^2-1}\left(t^{1/2} - t^{-1/2}\right)V_\infty(t), \tag{8}$$

with V_∞ denoting the Jones polynomial of L_∞, which is the link obtained by smoothing out a(ny) crossing in the room.

We consider now a diagram D in a braiding sequence of diagrams of genus $g(D) = \tilde{g}$ and some number of parameters $d \leq d_{\tilde{g}}$, where $d_{\tilde{g}}$ can be defined by (6). We have $d \geq 2\tilde{g}+1$ because of the $(2, 2\tilde{g}+1)$-torus knot diagram.

Then expand the relation (8) with respect to any of the d crossings, at which $\bar{t}_2$ moves can be applied, obtaining 2^d terms to the right. So their number in exponentially bounded in $\tilde{g}$, and hence it suffices to prove the inequality for each term separately.

Each term is of the form

$$\left(t^{1/2} - t^{-1/2}\right)^k \cdot V_L(t) \cdot t^{k'} \prod_{i=1}^{k} (1 + t^2 + \ldots + t^{2a_i}), \tag{9}$$

with $k \leq d$, $k' \in \mathbb{Z}$ and $\sum a_i = O(c(D))$, where $c(D)$ denotes the crossing number of D, and L being a link obtained by smoothing out (according to the usual skein rule) some set of crossings in the generating diagram. But the crossing number of L is linearly bounded in d, hence all its coefficients are exponentially bounded in d. Then, the coefficient sum of the product term is at most

$$\left[\frac{C\,c(K)}{d}\right]^d.$$

From this the theorem follows, as by [Ka2, Mu, Th] for an alternating diagram D of an alternating knot K, we have $c(D) = c(K)$, and by [Ga], $g(D) = g(K)$. □

Remark 3.2 C can be in principle written down explicitly. However, the resulting number so far has an unattractive magnitude. By [St], $d_{\tilde g} \le \dfrac{97\cdot 8^{\tilde g-2}-6}{7}$ for $\tilde g \ge 2$, but here it is possibly as well fertile to think about sharper bounds.

Another straightforward consequence was already noted in [St] and is repeated here, because it will be related to the extension of Morton's example.

Proposition 3.1 Let $t \in S^1 := \{\, z \in \mathbb{C} : |z| = 1 \,\}$. Then $\{\, V_K(t) : \tilde g(K) = g \,\} \subset \mathbb{C}$ is bounded for any $g \in \mathbb{N}$.

Proof. Repeat the previous formulas, noting that the partial sums of the Neumann series of t^2 and t^{-2} are both bounded if $|t| = 1$. $\square$

Finally, we come to the announced stability result for the "edges" of the Jones polynomial.

Definition 3.1 For some polynomial $V \in \mathbb{Z}[t,t^{-1}]$ define the minimal and maximal degree and the span (elsewhere called "breadth", not to the author's taste) of V by

$$\min\deg V := \min\{\, a \in \mathbb{Z} : [V]_{t^a} \ne 0 \,\}\,,\quad \max\deg V := \max\{\, a \in \mathbb{Z} : [V]_{t^a} \ne 0 \,\}\,,\ \text{and } \operatorname{span} V := \max\deg V - \min\deg V\,.$$

Then the list $\lambda_l V$ of V's leading coefficients of length l is the l-tuple $([V]_{t^{\min\deg V+k}})_{k=0}^{l-1} \in \mathbb{Z}^l$. Analogously define the list $\tau_l V$ of the trailing coefficients of V of length l.

Theorem 3.2 Fix g, l and n mod 2. Then the sets $\Lambda_{l,g} := \{\, \lambda_l V_K : K \in \mathcal{A}_{n,g} \,\}$ and $T_{l,g} := \{\, \tau_l V_K : K \in \mathcal{A}_{n,g} \,\}$ (with $\mathcal{A}_{n,g}$ as in (7)) stabilize as $n \to \infty$, that is, are all the same when $n \ge n_0$ for some n_0.

Proof. The proof of this property is closely related to its analoga for positive knots from [St3, §6]. We show it just for $\lambda_l V$ (because $\lambda_l V_K = \tau_l V_{!K}$, so in fact $\Lambda_{l,g} = T_{l,g}$).

By recalling carefully the proof of theorem 5.2 of [St3] for the case $t = 0$, we see that if $\{L_i\}_{i=-\infty}^{\infty}$ for i mod 2 fixed are links as in (8) (that is, a one-parameter antiparallel twist sequence), then $[V_{L_i}(t)]_{t^k}$, and more generally the $(l_2 - l_1 + 1)$-tuple $\big([V_{L_i}(t)]_{t^l}\big)_{l=l_1}^{l_2}$ for any $k, l_{1,2} \in \mathbb{Z}$, stabilize as $i \to \infty$, with the property, that a (not necessarily minimal) point of stabilization m_0, that is, a number, such that $[V_{L_{i_1}}(t)]_{t^k} = [V_{L_{i_2}}(t)]_{t^k}$ resp.

$$\big([V_{L_{i_1}}(t)]_{t^l}\big)_{l=l_1}^{l_2} = \big([V_{L_{i_2}}(t)]_{t^l}\big)_{l=l_1}^{l_2}$$

for all $i_{1,2} \ge m_0$, is dependent on k resp. $l_{1,2}$, but (very crucially) *independent* on the link diagram outside of the twist box.

We now have the following

Lemma 3.1 Let D be an alternating diagram and D' be obtained from D by applying a (antiparallel) twist at any of its positive resp. negative crossings. Then $\min\deg V(D) = \min\deg V(D')$ resp. $\max\deg V(D) = \max\deg V(D')$.

Proof. First forget about D's orientation and consider its unoriented version. It can be seen from the expression of $\min\deg V$ and $\max\deg V$ in terms of the checkerboard shading (see [Ad, pp. 160-162] or [Ka3]) that under a twist (in the unoriented version) $\min\deg V$ changes only locally, i. e., by something independent on the rest of the diagram.

Now, considering again D with orientation, $\min\deg V$ has a lower [St3, lemma 6.1] and upper [St6, theorem 4.2] bounds in terms of the diagram genus (which is fixed by an antiparallel twist) and the number of negative crossings (which is preserved as well, if the twist is at a positive crossing), hence $\min\deg V(D)$ ranges within some finite interval under antiparallel positive twists. But if the local change of $\min\deg V$ were non-zero, by applying successive further twists, we would be able to push $\min\deg V(D)$ arbitrarily high or low, contradicting one of the bounds.

Applying the argument on the mirror images, we get the statement for $\max\deg V$ and negative twists. $\square$

Remark 3.3 Therefore, twisting at positive crossings, $\min\deg V$ stays always the same. But then we see, that the dependence of m_0 on k resp. $l_{1,2}$ is in fact just a dependence on $k-\min\deg V$ resp. $l_{1,2}-\min\deg V$, because of the freedom to rescale V by a power of t (this is not very clear from the generating series representation of [St3, §6]). This is the second crucial point.

Prepared with lemma 3.1 and this observation, fix g, and consider separately any of the finitely many braiding sequences of alternating (knot) diagrams of genus g, and also consider therein all the twist boxes separately. First consider the twist boxes with positive crossings.

From lemma 3.1 and remark 3.3 we see that $\lambda_l V$ stabilizes after m_0 twists for some m_0 at any positive crossing (under further twists at that crossing), independently on how many twists have been done at the negative crossings. Therefore, to capture all contributions of knots in this braiding sequence to $\lambda_l V$, it suffices to consider separately the finitely many cases, where at each positive crossing at most m_0 twists are performed. Therefore, we fix for the rest of the proof the number of twists at each positive crossing.

We now show that the same argument can be made to apply for (twists at) the negative crossings.

Recall that (8) is the explicite form of the recursive relation

$$V_{k+4}(t) = (t^2+1)\,V_{k+2}(t) - t^2 V_k(t)\,, \tag{10}$$

with the subscripts of V denoting the number of positive (half-)twists. Now consider for a diagram D in the sequence

$$V'_D(t) := t^{c(D)}\,V_D(t)\,,$$

with $c(D)$ being the crossing number of D. Then because of $c(D_{k+2m}) = c(D_k)+2m$, V' again satisfies (10), but this time with subscripts of V denoting the number of negative twists. As D is alternating, by [Ka2, Mu, Th], $\min\deg V'(D) = -\min\deg V(!D)$, where $!D$ is the mirror image of D, and applying negative twists at D is the same as applying positive at $!D$, which by lemma 3.1 fixes $\min\deg V(!D)$, hence also $\min\deg V'(D)$.

Therefore, having fixed the number of twists at the positive crossings in D, we are interested in the leading l coefficients (that now have fixed positions) of the polynomials V' of the diagrams D, which again satisfy (10) in every twist box, the subscripts counting the number of negative twists. But because of (10), and its iterated version (8), these coefficients stabilize by the positive twist case argument. $\square$

Remark 3.4 Note that the use of [Ka2, Mu, Th] is crucial – we need upper control on $\min\deg V'(D)$, hence a lower control on the span of $V(D)$ from $c(D)$. The only (in fact, larger) class of knots, for which such control exists are the adequate knots of Lickorish and Thistlethwaite [LT]. It would be interesting, whether any of the results generalize to these knots. However, much trouble is expected because of the need of existence of an adequate diagram of minimal weak genus. On the other hand, from (8) it can be hoped, that a more careful analysis can prove the theorem 3.2 in full generality.

We conclude by another property of the Jones polynomials which is not expected to hold always, but at least "generically" with growing crossing number – the 2-periodicity almost everywhere of their coefficients. We just draw attention to the problem, leaving it open.

Definition 3.2 Call $[m,n]\subset[\min\deg V,\max\deg V]$ for some $V\in\mathbb{Z}[t,t^{-1}]$ and $m,n\in\mathbb{Z}$ with $n>m+2$ a 2-periodic interval of V, if $[V]_{t^k}=[V]_{t^{k+2}}$ for each $k\in[m,n-2]$. Denote this by $[m,n]\in 2p(V)$.

Conjecture 3.1

$$\frac{\sum\limits_{K\in\mathcal{A}_{n,g}}\Big|\bigcup\limits_{[m,n]\in 2p(V_K)}[m,n]\Big|}{n\,|\mathcal{A}_{n,g}|}\xrightarrow[n\to\infty]{}1$$

for any fixed g.

4. Inequalities for non-alternating knots

We show now a version of theorem 3.1 for non-alternating knots. An analogon to theorem 3.2 is a consequence of it.

Theorem 4.1 There is some constant $C > 0$ such that for any knot K and any $k \in \mathbb{Z}$ it holds

$$\big|\,[V_K(t)]_{t^k}\,\big| \,\le\, \big(C\operatorname{span} V_K\big)^{d_{\tilde g(K)}} \,\le\, \big(C\,c(K)\big)^{d_{\tilde g(K)}}\,.$$

Proof. If K has a diagram D in a d-parameter antiparallel braiding sequence of diagrams of genus $\tilde g(K)$ (so $d \le d_{\tilde g(K)}$), as before, from (8) you have that V_K is the sum of 2^d terms of the form (9), with $k' \in \mathbb{Z}$, $k \le d$ and $c(L) \le 2d$. Therefore, $V_K(t)\cdot(t+1)^d$ is the sum of terms as in (9), but this time with the product of $1-t^{2a_i+2}$, and so the coefficients of $V_K(t)\cdot(t+1)^d$ are bounded independently on $c(D)$ by something exponential in d. Now, w.l.o.g., multiply $\bar V_K(t) := V_K(t)\cdot(t+1)^d$ by a power of t, so that it to have minimal degree 0 (i. e., to be an honest polynomial it t with absolute term). The Taylor expansion of $\dfrac{1}{(t+1)^d}$ around $t=0$ has an n-th coefficient, which is $O(n^{d-1})$ in n, with $O(\,.\,)$ independent on d. Therefore, $\big[\bar V_K(t)\cdot\dfrac{1}{(t+1)^d}\big]_{t^k} = O(k^d)$ in k with $O(\,.\,)$ depending exponentially in d. But clearly $\big[\bar V_K(t)\cdot\dfrac{1}{(t+1)^d}\big]_{t^k} = 0$ for $k > \operatorname{span} V_k$, so the first assertion follows. The second inequality follows from [Ka2, Mu, Th]. $\square$

Corollary 4.1 $\{\lambda_l V_K : \tilde g(K) = g\}$ is finite for any l and g.

Proof. Use the bijection between $\lambda_l V_K$ and $\lambda_l \bar V_K$, and prove the assertion for $\lambda_l \bar V_K$. $\square$

A more detailed study may also show a stabiliy property of some kind, for example, when $\operatorname{span} V_K \to \infty$.

Corollary 4.2

$$\max_{\tilde g(K)=g} \big|\,[V_K(t)\cdot(t+1)^{d_g}]_{t^k}\,\big| \le C^g$$

for some constant C independent on k, K and g, that is, $V_K(t)\cdot(t+1)^{d_g}$ has bounded coefficients over all K with $\tilde g(K) = g$, and moreover the number of non-zero coefficients of $V_K(t)\cdot(t+1)^{d_g}$ is also bounded for fixed g. $\square$

Corollary 4.3 For K positive we have

$$\operatorname{span} V_K \,\ge\, C_{g(K)}\ \sqrt[2d_{g(K)}+1]{c(K)} - 1 \tag{11}$$

for some constant C depending on $g(K)$. In particular, the are only finitely many positive knots with Jones polynomial of given minimal and maximal, or just maximal, degree.

Proof. Use the inequality [St7, theorem 6.1] for $v_2 = -{}^1\!/_6 V''(1)$. $\square$

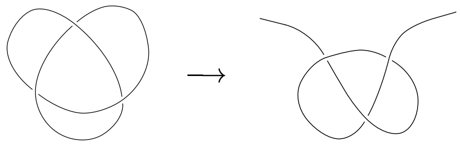
Remark 4.1 In (11), $\operatorname{span} V_K + 1$ may stronger be replaced by the number of non-zero coefficients of V_K, and $c(K)$ by the maximal crossing number of a positive reduced diagram of K.

Corollary 4.4 (see conjecture 9.1 of [St]) Among the Jones polynomials of knots of given $\tilde g$, only finitely many polynomials of given span occur.

Proof. By theorem 4.1, Jones polynomials of knots of given $\tilde g$ with given span have only finitely many coefficient lists between minimal and maximal degree. But (for *knots*, unlike for links) the coefficient list recovers the minimal degree (amd hence the polynomial), because $V(1) = 1$ and $V'(1) = 0$. $\square$

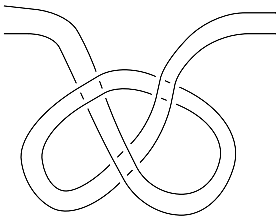
5. Genus and weak genus

Definition 5.1 The *untwisted double tangle* of a knot is obtained by cutting the knot diagram



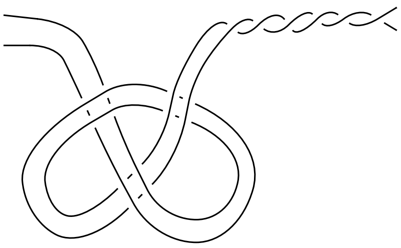
, (12)

replacing each strand by two



, (13)

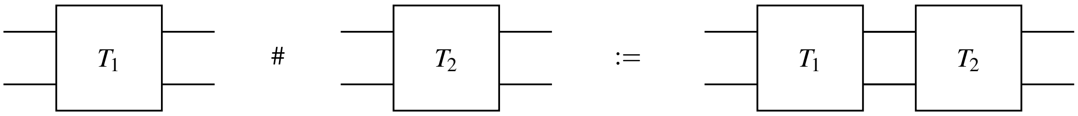
and adding a number of half-twists, which are doubly as many as the writhe of the knot diagram (12), and are positive when orienting the strands antiparallelly



(with the usual convention that -1 half-twist is a half-twist with the crossing changed). A tangle obtained by any other number of half-twists is called *twisted* double tangle of the knot. The difference of the number of its half-twists and the number of half-twists of the untwisted double tangle is called the *twist* of the twisted double tangle.

Let $w_\pm$ be the tangles  and  .

Definition 5.2 The sum $T_1\#T_2$ of two tangles T_1 and T_2 is defined by



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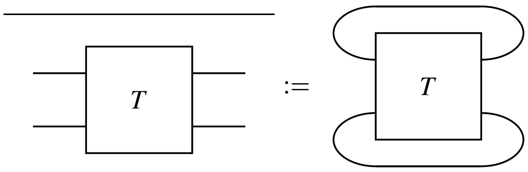
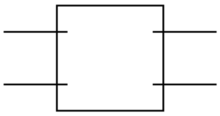
The closure $\overline{T}$ of a tangle T is defined by



Theorem 5.1 If T is a double tangle and some of the knots $\overline{\#^n T\#w_\pm}$ has a Jones polynomial, in which there are (at least) two coefficients with absolute value 3 or (at least) one coefficient with absolute value at least 4, or (at least) six coefficients with absolute value 1, then $\tilde g(\overline{\#^n T\#w_\pm}) \xrightarrow[n\to\infty]{} \infty$ (while clearly all $\overline{\#^n T\#w_\pm}$ are doubled knots and hence have genus one).

Proof. Assume that $K_n := \overline{\#^n T \# w_\pm}$ have bounded $\tilde{g}$. By theorem 4.1, our strategy will be to find some $k_n \in \mathbb{Z}$, for which $[V_{\overline{\#^n T \# w_\pm}}(t)]_{t^{k_n}}$ grows exponentially in n, unless the assertion is satisfied. First, we use the Kauffman [Ka] definition for V and replace V by the Kauffman bracket $\langle\,.\,\rangle$ (as all the normalization does not affect the norm of an evaluation on any point on S^1 and changes the coefficients just by a sign).

Then consider T in the Kauffman bracket skein module of



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We have therein

$$T = P_1'(A) \qquad + P_2'(A) \;)\,($$

for some $P_{1,2}' \in \mathbb{Z}[A, A^{-1}]$. Then by straightforward calculation

$$\#^n T = \frac{1}{-A^2 - A^{-2}} \left[\left(P_2'(-A^2 - A^{-2}) + P_1' \right)^n - P_1'^n \right])\,(+ P_1'^n$$

and hence

$$B_n := \langle \overline{\#^n T \# w_\pm} \rangle = \frac{1}{-A^2 - A^{-2}} \left[\left(P_2'(-A^2 - A^{-2}) + P_1' \right)^n - P_1'^n \right] \left\langle \quad \right\rangle + P_1'^n \left\langle \quad \right\rangle \cdot A^{k_1}\,.$$

Therefore, using $\left\langle \quad \right\rangle = -A^k(1 + A^8)$ for some $k \in \mathbb{Z}$ and $\left\langle \quad \right\rangle = 1$, we get, normalizing B_n by a power of A,

$$B_n = A^k \frac{1 + A^8}{1 + A^4} \left[[P_2'(-A^2 - A^{-2}) + P_1']^n - P_1'^n \right] + P_1'^n = \left(\frac{A^k + A^{k+8}}{1 + A^4} \right) P_2^n + \left(1 + \frac{-A^k - A^{k+8}}{1 + A^4} \right) P_1^n, \qquad (14)$$

with $P_1 := P_1'$ and $P_2 := P_2'(-A^2 - A^{-2}) + P_1'$.

The shape of B_n is exponential, and we attack it using the following elementary function theoretic lemmas.

Lemma 5.1 Let $f \in \mathbb{Z}[A, A^{-1}]$. If f, regarded as a function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, has the property $\max\limits_{S^1} |f| \le 1$ and $f \ne 0$, then $f = \pm A^k$ for some $k \in \mathbb{Z}$.

Proof. Use the relation

$$\sum_{i=-\infty}^{\infty} [f]_{A^i}^2 = \int_0^1 \left| f\left(e^{2\pi i u}\right) \right|^2 du\,. \qquad \square$$

Lemma 5.2 Let $f : \mathbb{C} \setminus S \to \mathbb{C}$ be a holomorphic function for some finite set $S \ni 0$, with $f(\bar{x}) = \overline{f(x)}$ (where bar denotes conjugation). If then f maps some infinite subset of $S^1 \subset \mathbb{C}$ to S^1, then $f(x)f(1/x) \equiv 1$ wherever defined.

Proof. Use that $f(x)f(1/x)$ is a holomorphic function wherever defined and is equal to 1 on a set with a convergence point. $\square$

The rest is basically applying appropriately these lemmas.

Lemma 5.3 For any two polynomials P_1 and P_2 in $\mathbb{Z}[A, A^{-1}]$ with $P_1 \ne \pm A^{k_1}$ or $P_2 \ne \pm A^{k_2}$ for any $k_{1,2} \in \mathbb{Z}$, there are infinitely many $A \in S^1$ with $\left| \frac{A^k + A^{k+8}}{1+A^4} \right| \ne \left| \frac{1 + A^4 - A^k - A^{k+8}}{1+A^4} \right|$ and $\max(|P_1(A)|, |P_2(A)|) > 1$.

Proof. Assume that $P_1 \neq \pm A^{k_1}$ or that $P_2 \neq \pm A^{k_2}$. As the assertion is symmetric w.r.t. P_1 and P_2, assume w.l.o.g., that $P_1 \neq \pm A^{k_1}$. Then by lemma 5.1, there is some $x \in S^1$ and $\varepsilon > 0$ such that

$$|P_1|\Big|_{S^1\cap B(x,\varepsilon)} > 1\,,$$

with $B(x,\varepsilon)$ being the ball around x with radius ε. Set $X_1(A) := A^k + A^{k+8}$ and $X_2(A) := 1 + A^4 - A^k - A^{k+8}$. If now, for infinitely many $A \in S^1$ we have $|X_1(A)| = |X_2(A)|$, then applying lemma 5.2 on X_1/X_2 outside of the zeros of X_2 and $1+A^4$, we see that $X_1(A)X_1(1/A) = X_2(A)X_2(1/A)$, in particular every zero $A \neq 0$ of X_1 must be either a zero of X_2 or the inverse of such. However, $e^{\pm\pi i/8}$ are zeros of X_1, but not of X_2.

Therefore, all but finitely many $A \in S^1$ satisfy $|X_1(A)| \neq |X_2(A)|$, in particular almost all A in $S^1 \cap B(x,\varepsilon)$. □

But now, continuing the proof of theorem 5.1, if $\max(|P_1(A)|,|P_2(A)|) > 1$ and $\left|\frac{A^k+A^{k+8}}{1+A^4}\right| \neq \left|\frac{1+A^4-A^k-A^{k+8}}{1+A^4}\right|$ for some A not zero of $X_{1,2}$ and $1+A^4$, then for

$$A \in \{\, x \in S^1 \,:\, |P_1(x)| > 1 \wedge 0 \neq |X_1(x)| \neq |X_2(x)|\,\} \setminus \{ \text{ zeros of } 1+A^4 \,\}$$

we have

$$|B_n(A)| \geq \frac{1-\varepsilon}{|1+A^4|}\min\big(\,|X_1(A)|,\,|X_2(A)|,\,\big|\,|X_1(A)|-|X_2(A)|\,\big|\,\big)\,\max\big(|P_1(A)|,|P_2(A)|\big)^n \xrightarrow[n\to\infty]{} \infty,$$

with exponential growth. But from (14), comparing the orders of zeros or poles of B_n as $A \to 0$ and $A \to \infty$ (or from [Ka2, Mu, Th] using the evident fact that $c(\overline{\#^n T \# w_\pm}) = O(n)$), we see that $\operatorname{span} B_n$ is linearly bounded in n, which means that some coefficients of B_n grow exponentially in n, and we would be done by contradiction to theorem 4.1.

Therefore, P_1 and P_2 are monomials with coefficients ± 1 (as happens when T is an unknot double, i. e., $\overline{\#^n T \# w_\pm}$ are twist knots). But then the only possibility for them, so as $B_n \in \mathbb{Z}[A,A^{-1}]$ for any n, is to differ by a power of A^4, so that

$$B_n = A^k(1+A^8)\big(1-A^4+A^8-\,\dots\,\pm A^{4(np-1)}\big) + A^{nl}$$

for some $k,p,l \in \mathbb{Z}$, from which the claim is evident. □

We chose to use the new property of §4 in a part of our proof, although it can also be done in alternative ways. We invite the reader to think about them.

Exercise 5.1 Use (14) to show that if $P_{1,2} \neq \pm A^{k_{1,2}}$, then $\lambda_l B_n$ or $\tau_l B_n$ are infinitely many for some value of l, without using the lemmas, so that the conclusion $P_{1,2} = \pm A^{k_{1,2}}$ is also possible using corollary 4.1. In a much easier (and less interesting) way, deduce the same conclusion also from proposition 3.1.

We give some hints to the reader, giving a rough sketch of the argument.

As λ_l is mapped bijectively under multiplication by a fixed polynomial, eliminate the denominators in (14), and consider

$$P_n := \underbrace{\left(1+A^4-A^k-A^{k+8}\right)}_{X_1(A)} P_1^n(A) + \underbrace{\left(A^k+A^{k+8}\right)}_{X_2(A)} P_2^n(A)\,. \tag{15}$$

Lemma 5.4 Let the minimal order of V be defined by $\min\operatorname{ord} V := \min\{\, m > 0 \,:\, [V]_{t^{\min\deg V+m}} \neq 0\,\}$, if $V \in Z[t,t^{-1}]$ is not a monomial. Then

$$[V^n(t)]_{t^{\min\deg V+k}} = O\left(n^{\lfloor k/\min\operatorname{ord} V\rfloor}\cdot [V]^n_{t^{\min\deg V}}\right)$$

and

$$[V^n(t)]_{t^{\min\deg V+k\min\operatorname{ord} V}} = o\left(n^k\cdot [V]^n_{t^{\min\deg V}}\right)\,.$$ □

An analogous definition and statement hold for the maximal order $\max\operatorname{ord} V$ of V.

From (14) clearly not both P_1 and P_2 are zero, and if just one is zero, the other cannot be a single monomial, so we would be done by the above lemma. Therefore, assume that both P_1 and P_2 are non-zero.

Now, if $\min\deg P_2 < \min\deg P_1$ and $P_2 \neq \pm A^k$, then for any l for sufficiently large n we have $\lambda_l P_n = \lambda_l\big(X_2 P_2^n\big)$, which by the lemma for l sufficiently large has an unboundedly growing coefficient. An analogous argument with P_1 and the maximal degree shows that we are done unless $\min\deg P_1 = \min\deg P_2$ and $\max\deg P_1 = \max\deg P_2$. Then the lemma shows that $\min\operatorname{ord} P_1 = \min\operatorname{ord} P_2$ and $\max\operatorname{ord} P_1 = \max\operatorname{ord} P_2$ by a similar argument. For example, if $\min\operatorname{ord} P_1 < \min\operatorname{ord} P_2$, consider the coefficient of $t^{\min\deg(X_1P_1^n)+\min\operatorname{ord} P_1\cdot l}$ for l sufficiently large, and compare in (15) the growth rates of this coefficient as $n\to\infty$ for $X_1P_1^n$ and $X_2P_2^n$. But now a contradiction follows, considering the above mentioned coefficient for $l=1$, because at least one of $\min\deg X_1 \neq \min\deg X_2$ or $\max\deg X_1 \neq \max\deg X_2$ holds, no matter what k is. Therefore, $P_{1,2}$ must be monomials, and then clearly they must have coefficient ± 1.

Example 5.1 The two 14 crossing (twisted) doubles of the left-hand trefoil with positive and negative clusp, in Thistlethwaite's tables (see [HTW]) included as 14_{35575} and 14_{41716}, have the Jones polynomials (in the notation of [St5])

$$V(14_{35575}) = -1\ 1\ -1\ 1\ 0\ 1\ [0]\ 1\ -1\ 1\ -2\ 1 \quad \text{and} \quad V(14_{41716}) = 1\ -1\ 1\ -1\ [1]\ -1\ 1\ -1\ 1\ -1\ 2\ -1\,,$$

and hence theorem 5.1 can be applied to the tangle in (13) with both w_+ and w_-.

6. Problems

In fact, the motivation for this note was to develop the results of §3 so far as to do the construction of §5 without use of proposition 3.1. Using this proposition, the proof is somewhat simpler, but it appeared nicer to link both parts of the paper in the chosen way.

Nevertheless, I prefer to conclude stressing again some problems that suggest to be of some significance within the framework of this note.

Problem 6.1 Is there an upper bound for $\big|[V_K(t)]_{t^k}\big|$ for K alternating in terms of $g(K)$ and possibly k, but not $c(K)$ (as there is for positive K)? Is there a similar inequality also for non-alternating knots?

Problem 6.2 Can one generalize theorem 3.2 to a stability property for arbitrary knots (and weak genus) by more refined study of (8)?

Problem 6.3 Are the only finitely many positive knots with Jones polynomial of given span? If we had an infinite series of such knots, then by corollary 4.3 we know that their genera must grow unboundedly, but yet we cannot exclude such a case.

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