ON THE DEFINITION OF GRAPH INDEX

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Abstract. This paper discusses a flaw in Murasugi-Przytycki's Memoir "An index of a graph with applications to knot theory" Mem. Amer. Math. Soc. **106** (1993). We point out and partly fix a gap occurring in the proof of Murasugi-Przytycki's braid index inequalities involving the graph index. We explain why their notion of index fails to precisely reflect the reduction of Seifert circles by their diagram move, and redefine the index to account for that discrepancy.

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1 Introduction

As an important part of the literature devoted to studying the braid index in the aftermath of the discovery of the Jones polynomial and its successors, Murasugi-Przytycki's Memoir [MP] introduces the notion of index of a graph.

Their motivation stems from the relation discovered [Ya] between the braid index and the number of Seifert circles of a link diagram. They introduce a link diagram move to reduce the number of Seifert circles (Figure 8.2 of [MP]).

The Murasugi-Przytycki move reduces this number by one, and can (often) be applied repeatedly. However, the choice of move(s) is in general highly ambiguous, and the number of applicable moves depends heavily on this choice. The highest economy on Seifert circles is thus achieved when the number of moves is maximized.

Murasugi-Przytycki's definition of graph index bases on a transformation of graphs, which is to model their diagram move on the level of Seifert graphs (see the proof of lemma 8.6 in [MP]). The precise definition of the index (and where a problem occurs with it) will be discussed in detail below, but it is important to notice already here that it entails a maximization over possible ways to modify the graph.

Murasugi-Przytycki aim to obtain then the inequality

$$b(L) \leq s(D) - \mathrm{ind}(D)\,. \tag{1}$$

Here D is a diagram of $s(D)$ Seifert circles of a link L with braid index $b(L)$, and $\mathrm{ind}(D)$ is the index of the Seifert graph of D. The inequality (1) is one of the central results of [MP], and many applications there and elsewhere, for example [Oh], rely on it.

During our study of Murasugi-Przytycki's proof, we found a gap. It occurred when we wanted to understand the diagram move of Figure 8.2 of [MP]. Murasugi-Przytycki seem to assume that Figure 8.2 is the general case, but we will explain that it is not. Taking care of the missing cases leads to a modified definition of index, which we call ind_0. Roughly speaking, the correction needed is that in certain situations some edges in the star of a vertex are not contracted (unlike in their procedure; see below definition 3.2). Therefore, Murasugi-Przytycki's diagram move just proves instead of (1) that

$$b(L) \le s(D) - \operatorname{ind}_0(D)\,. \tag{2}$$

Then the question naturally arises of how $\operatorname{ind}(D)$ and $\operatorname{ind}_0(D)$ relate to each other. We will argue that

$$\operatorname{ind}(D) \le \operatorname{ind}_0(D)\,, \tag{3}$$

which justifies (1). This is necessary in order to rehabilitate the applications of this inequality inside and outside Murasugi-Przytycki's Memoir. We then speculated, based on our computational evidence, whether in fact always

$$\operatorname{ind}(D) = \operatorname{ind}_0(D)\,. \tag{4}$$

Later Traczyk [Tr] provided an argument that this is indeed true, by proving the reverse inequality to (3). Our understanding is that Traczyk's work thus clarifies an important point in the matter, but that this is not exactly what is needed (and it is not enough) to fix the error.

In that realm, the explanation of ind_0 and (3) remains necessary, and succeeds only at a (minor) cost. This drawback is that Murasugi-Przytycki's definition of index loses its geometric meaning *per sé*. It simplifies the true transformation of the Seifert graph under their diagram move, in a way which is *a priori* incorrect but (fortunately) *a posteriori* turns out to still give the right quantity. This fact must be taken care of in subsequent applications of Murasugi-Przytycki's method, e.g., in [MT].

The following account tries to explain the details.

2 Braid representations and braid index

The *braid group* B_n on n strands (or strings) is considered to be generated by the Artin *standard generators* σ_i for $i = 1, \dots, n-1$. These are subject to relations of the type $[\sigma_i, \sigma_j] = 1$ for $|i-j| > 1$, which we call *commutativity relations* (the bracket denotes the commutator) and $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$, which we call *Yang-Baxter* (or shortly YB) relations.

A classical theorem of Alexander [Al] asserts that each link L can be represented as $L = \hat{\beta}$, the *closure* of some braid β. We call β then a *braid representation* of L. The *braid index* $b(L)$ of a link L is the smallest number of strands among all braid representations of L. See [Mo, FW, Mu]. A braid β on $b(L)$ strands with $\hat{\beta} = L$ is called a *minimal braid (representation)* of L.

The *skein polynomial* P [F&, LM] is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$l^{-1}P\Big($  $\Big) + l\,P\Big($  $\Big) = -m\,P\Big($  $\Big)\,.$ (5)

As usual, the three fragments depict link diagrams identical elsewhere. The convention uses the variables of, but differs from the one adopted in, [LM] by the interchange of l and l^{-1}.

Let $P \in \mathbb{Z}[l^{\pm1}, m^{\pm1}]$. The *minimal* resp. *maximal l-degree* $\min\deg_l P$ resp. $\max\deg_l P$ is the minimal resp. maximal exponent of l in a monomial (with non-zero coefficient) in P. Let $\operatorname{span}_l P = \max\deg_l P - \min\deg_l P$.

A crossing as on the left in (5) has *writhe* (or *skein sign*) 1 and is called *positive*. A crossing as in the middle of (5) has writhe -1 and is called *negative*. The *writhe* $w(D)$ of a link diagram D is the sum of writhes of all its crossings.

The replacement of a (positive or negative) crossing by the the rightmost picture in (5) is called *smoothing out*. When all crossings of D are smoothed out, we have a collection of loops called *Seifert circles*. Under undoing the smoothing operation, we can regard each crossing in D as connecting two (distinct) Seifert circles. Let $s(D)$ be the number of Seifert circles of D.

In [Mo, FW] it was proved for the skein polynomial $P(l,m)$ that

$$\frac{1}{2}\operatorname{span}_l P(L) + 1 \le b(L)\,, \tag{6}$$

the *Morton-Williams-Franks* (MWF) *inequality*.

It was soon noticed that for many links the Morton-Williams-Franks inequality is sharp (i.e., an equality), and for a while it had been conjectured that this would be so for all alternating links. Murasugi-Przytycki disproved this conjecture, obtaining (among others) an 18 crossing counterexample alternating knot.

The inequality (6) results from two other inequalities, due to Morton, namely that for a diagram D, we have

$$1-s(D)+w(D) \le \min\deg_l P(D) \le \max\deg_l P(D) \le s(D)-1+w(D)\,. \tag{7}$$

Williams-Franks showed these inequalities for the case of braid representations. Later it was observed from the algorithms of Yamada [Ya] and Vogel [Vo] that the braid version is actually equivalent to, and not just a special case of, the diagram version. (These algorithms allow to turn any diagram D into a braid diagram without altering $s(D)$ and $w(D)$.) Nonetheless we will refer below to (7) as 'Morton's inequalities'.

These inequalities were later improved in [MP] in a way that allows to settle the braid index problem for many links (see theorem 7.1 or also [Oh]). For this purpose, Murasugi-Przytycki developed the concept of index of a graph. We recall some main points of Murasugi-Przytycki's work, referring to [MP] for further details.

3 Graph index

Graphs will be finite. It will be no restriction to assume that they are planar, i.e., admitting a planar embedding. (We later make a remark on the ambiguity of the planar embedding.) We allow different edges to connect the same two vertices. Such edges will be counted (and in certain cases, treated) separately. We thus understand a *multiple edge* as a set consisting of the edges connecting the same two vertices.

An edge is *simple* if no other edge connects the same two vertices. Such an edge will be sometimes denoted by its two vertices (order irrelevant).

Loop edges can also be allowed, but are not very relevant.

Definition 3.1 Let G be a connected graph. For a vertex v in G let the *star* $\operatorname{star} v$ of v be the set of edges in G *incident* from v, i.e., those for which v is one of the endpoints. Let $G_v = G/v$ be the graph obtained from G by contracting $\operatorname{star} v$.

Let $G\setminus v$ be the graph obtained from G by deleting $\operatorname{star} v$, and *additionally* v itself. (When we delete an edge, we understand that any vertex it is incident to is *not* to be deleted along the way.) We call v a *cut vertex* of G, if $G\setminus v$ is disconnected.

Let G be a connected *signed* graph; 'signed' will mean for us that each edge carries a sign $+$ or $-$. There is no problem in extending the various introduced graph operations to signed graphs.

Definition 3.2 We define (recursively) a sequence of edges $\mu=(e_1,\dots,e_n)$ to be *independent* in a graph G, if the following conditions are satisfied.

1. The empty (edge) sequence is independent per definition.
2. Let e_1 connect vertices v_1 and v_2. Then we demand that e_1 is simple, i.e. there is no other edge connecting $v_{1,2}$, and that $e_2,\dots,e_n$ is independent in (one of) G_{v_1} or G_{v_2} (i.p., $\{\,e_2,\dots,e_n\,\}$ is disjoint from at least one of $\operatorname{star} v_1$ and $\operatorname{star} v_2$, resp.).

An *independent set* is a set of edges admitting an ordering as an independent sequence.

The *index* $\operatorname{ind}(G)$, resp. *positive index* $\operatorname{ind}_+(G)$ and *negative index* $\operatorname{ind}_-(G)$ of G are defined as the maximal length of an independent edge set (or sequence), resp. independent positive or negative edge set/sequence in G. A sequence is *maximal independent* if it realizes the index of G.

Now to each link diagram D we associate its *Seifert graph* $G = \Gamma(D)$, which is a planar bipartite signed graph. It consists of a vertex for each Seifert circle in D and an edge for each crossing, connecting two Seifert circles. Each edge is signed by the writhe (or skein sign) of the crossing it represents, as explained below (5). Note that several edges between the same vertices can thus occur, and they may also carry different signs. We will for convenience sometimes identify crossings/Seifert circles of D with edges/vertices of G.

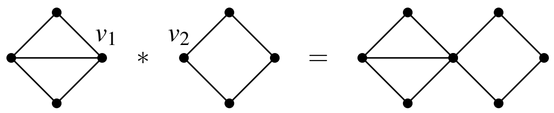
Then we can set $\operatorname{ind}_{(\pm)}(D) = \operatorname{ind}_{(\pm)}(\Gamma(D))$. Murasugi-Przytycki claim the following:

Proposition 3.1 (see [MP, (8.4) and (8.8)]) If D is a diagram of an oriented link L, then

$$\begin{aligned} \max\deg_l P(L) &\le w(D) + s(D) - 1 - 2\operatorname{ind}_+(D) && (8) \\ \min\deg_l P(L) &\ge w(D) - s(D) + 1 + 2\operatorname{ind}_-(D) && (9) \\ b(L) &\le \operatorname{mpb}(D) := s(D) - \operatorname{ind}(D)\,. && (10) \end{aligned}$$

An important operation on diagrams studied in relation to the index is this of *Murasugi sum decomposition* (see [Cr, §1]). On the level of Seifert graphs it corresponds (mainly) to block decomposition.

Definition 3.3 The *join* (or *block sum*) $G_1 * G_2$ of two graphs G_1 and G_2 is defined by



(11)

This operation depends on the choice of a vertex v_i in each one of the graphs G_i. (Although this dependence will not be notationally highlighted, it should be kept in mind.)

Every connected non-trivial (i.e. with at least one edge) graph G can be written as a join $G_1 * \ldots * G_n$ for some non-trivial connected graphs G_i, such that no G_i has a cut vertex. We call G_i the *block components* or *join factors* of the graph G.

The precise relation between block and Murasugi sum decomposition is as follows. A Seifert circle in D is *separating*, if it has crossings attached to it from both its interior and exterior. A diagram with no separating Seifert circles is called *special*. The *blocks* of D are the connected sum components of the pieces of D obtained by Murasugi sum decomposition of D along its separating Seifert circles. See for this [Cr, §1], but keep in mind that Cromwell's definition of blocks does not take into account connected sum decomposition. (Thus some of Cromwell's blocks can decompose into several blocks in our sense.)

Then *each block component of* $\Gamma(D)$ *is the Seifert graph of a block of* D. (It is for the sake of this analogy that we alter here Cromwell's definition.)

If $G = \Gamma(D)$ is a block component itself (i.e., has no cut vertex, and D is prime and special), one can recover D uniquely from a concrete planar embedding of G (both regarded up to moves in S^2). It is helpful, e.g., in comparing figures 1 and 2 below, to keep in mind the correspondence between a block of D and the planar embedding of its Seifert graph. However, under block sum, there is little sense in dwelling upon planar embeddings. This is why the block sum (11) of two graphs G_1 and G_2 is understood to depend on not more than the choice of vertices v_i in G_i.

For any diagram D, we have

$$\operatorname{ind}_+(D) + \operatorname{ind}_-(D) \ge \operatorname{ind}(D)\,. \qquad (12)$$

For alternating (and more generally homogeneous [Cr]) diagrams D equality holds, because each join factor of $\Gamma(D)$ contains only edges of the same sign. This implies that if in such diagrams (8), (9) are sharp, then (6) and (10) become sharp, too.

Conjecture 3.1 (Murasugi-Przytycki) If D is an alternating diagram of a link L, then $b(L) = \operatorname{mpb}(D)$.

4 Hidden Seifert circle problem

Now we must understand the move of Murasugi-Przytycki that corresponds to the choice of a simple edge e and the contraction of the star of v in G. To set the record straight, we should mention that this move was considered, apparently simultaneously and independently, also by Chalcraft [Ch], although merited there only with secondary attention. With this understanding, we will refer to it below still as the Murasugi-Przytycki move.

This move is shown in figure 8.2 of [MP], but see figure 1 below for a more authentic situation. Let D be the diagram before the move and D' the diagram resulting from it. Let us for simplicity identify an edge with its crossing and a vertex with its Seifert circle (see the remark above proposition 3.1). In this language, the move of Murasugi-Przytycki eliminates one crossing, corresponding to e. The crossings of the other edges $e' \neq e$ incident to v do not disappear under the Murasugi-Przytycki move. Instead, they become in D' parts of join factors of $\Gamma(D')$ that correspond to a Murasugi summand on the opposite side of the modified Seifert circle. See the proof of lemma 8.6 in [MP], and figures 1 and 2 below.

The subtlety, which seems to have been overlooked in the proof of [MP], is illustrated in figures 1 and 2. The Seifert circles adjacent to v may be nested in D in such a way that relaying the arc of v by the move, one does (and can) *not* go along *all* Seifert circles adjacent to v. In the Seifert graph $G' = \Gamma(D')$ of D' some of the edges incident to v in $G = \Gamma(D)$ may not enter, as written in the proof of lemma 8.6 in [MP], into block components that are 2-vertex graphs (with a multiple edge).

Still we see that contracting the star of v in $G = \Gamma(D)$, we obtain a graph $\tilde{G} = G/v$, which is a *contraction* of $G' = \Gamma(D')$. (We will later describe *exactly* how G' is constructed from G, but let us for the time being use the easier to obtain G/v instead.) Here contraction $\tilde{G}$ of a graph G' means that $\tilde{G}$ is obtained from G' by contracting some (possibly several or no) edges, and we allow multiple edges in G' to be contracted (by doing so simultaneously with all edges they consist of).

More precisely, the difference between the block component of $\tilde{G}$ and G' is that in the last block component of G' in figure 2 the star of v is contracted to obtain the block component of $\tilde{G}$ consisting of edges h and k. So for the proof of lemma 8.6 in [MP] and (10), we actually need the following lemma.

Lemma 4.1 If a graph H' is a contraction of H, then $\operatorname{ind}(H') \leq \operatorname{ind}(H)$.

Proof. We prove that each independent set of edges in H' is independent in H. We do this inductively over the number of vertices of H.

Let $e_1,\dots,e_n$ be independent in H'. Then $e_2,\dots,e_n$ are independent in H'/v_1, for some vertex v_1 to which e_1 is incident. All edges $e_1,\dots,e_n$ exist in H, and so does v_1. (During the contractions that turn H into H', the vertex v_1 in H may be identified with others.)

Now H'/v_1 is a contraction of H/v_1, and thus by induction assumption, $e_2,\dots,e_n$ are independent in H/v_1. Moreover, since e_1 is simple in H', it is simple in H. (Contractions cannot eliminate multiple edges except by contracting them.) Thus, $e_1,\dots,e_n$ are independent in H, as we wanted. □

With the lemma, we will establish in lemma 5.1 below that we can reduce *at least* $\operatorname{ind}(D)$ Seifert circles by Murasugi-Przytycki, moves, and (10) is recovered.

Still is should be understood that the contraction of a vertex is not fully correct as modelling the Murasugi-Przytycki diagram move.

5 Modifying the index

It becomes necessary to understand exactly the transformation of the Seifert graph G under the move of Murasugi-Przytycki. We describe it now, also filling in the detail overlooked by them.

Now we use *marked graphs*. This means that edges have a $\mathbb{Z}_2$-graduation. Each edge either carries a *mark* (and is *marked*) or not (and is *unmarked*). Its *marking* is the status according to this graduation. This distinction is

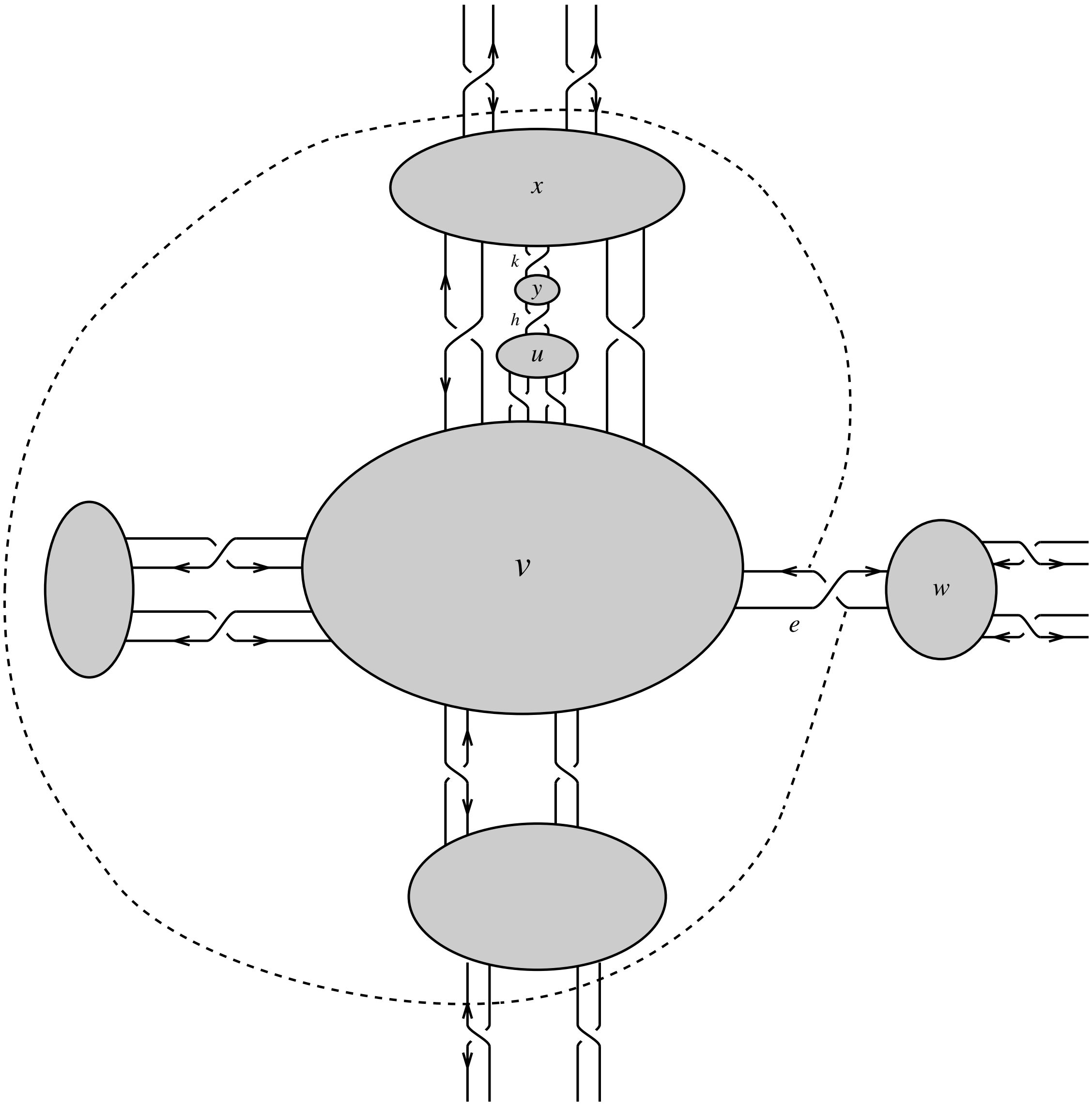


Figure 1: A move of Murasugi-Przytycki, where the relayed strand (dotted line) does not go along a Seifert circle (denoted as u) adjacent to v. The Seifert circles are depicted in gray to indicate that their interior may not be empty.

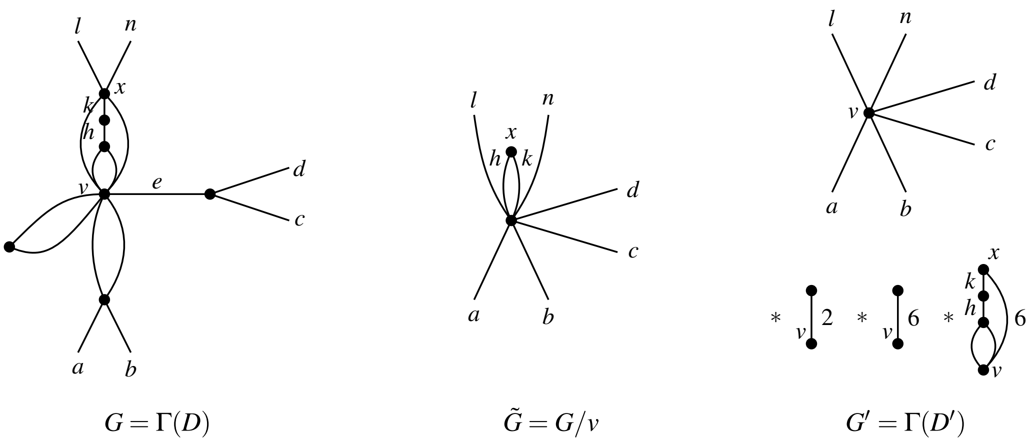


Figure 2: The various Seifert graphs of the diagrams related to the move of Murasugi-Przytycki in figure 1, in the case when the relayed strand does not go along all Seifert circles adjacent to v. The graph of D' is given in its block decomposition, which corresponds to the Murasugi sum decomposition along the newly created Seifert circle. For simplicity, we display a multiple edge by attaching the multiplicity to the edge drawn as simple (otherwise, a letter attached just indicates the name).

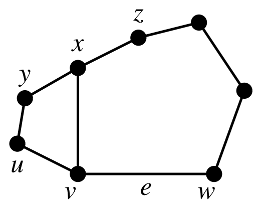
different from the $+/-$ signing. However, marked edges carry no sign, so that when signs are relevant, one should distinguish edges into positive, negative (unmarked) and marked ones.

We assume for the rest of the exposition that *G is bipartite*. Thus G has no loop edges (isthmusses) and no cycles of length 3, which avoids some technical difficulties.

In the initial (Seifert) graph all edges are unmarked (and carry, if relevant, the sign of their corresponding crossing). A marked edge is to be understood as one that cannot be chosen as an edge e. It corresponds to a multiple edge.

Definition 5.1 We choose a non-marked edge e and a vertex v of e. Let w be the other vertex of e (see figure 1). We define the notion *on the opposite side to e* as follows.

A vertex $y \neq v, w$ is on the opposite side to e if there is a vertex $x \neq v, w, y$ adjacent to v such that y and w are in different connected components of $(G \setminus v) \setminus x$.



(13)

(Here '$\setminus$' stands for the deletion of a vertex together with all its incident edges – but not its adjacent vertices; *cf.* definition 3.1.) Otherwise we say y is *on the same side as e*.

The meaning of this distinction is that the Murasugi-Przytycki move lays the arc along a Seifert circle x adjacent to (the Seifert circle of) v, if x is on the same side as e. This move affects the crossings that connect x to v, or to a Seifert circle z on the same side as e.

Definition 5.2 Let G be a marked graph, v a vertex of G, and e a *simple unmarked* edge between v and another vertex w. We define now the marked graph $G/_e v$.

The vertices of $G/_e v$ are those of G except w. The edges and their markings are chosen by copying those in G as follows. Let an edge e' in G connect vertices $v_{1,2}$.

Case 1. v is among $v_{1,2}$, say $v = v_1$.

Case 1.1. If the other vertex v_2 of e' is w (i.e. $e = e'$), then e' is deleted.

Case 1.2. If v_2 is on the opposite side to e, then e' is retained in $G/_e v$ with the same marking.

Case 1.3. If v_2 is on the same side as e, then e' is retained in $G/_e v$, but marked.

Case 2. v is not among $v_{1,2}$.

Case 2.1. If none of $v_{1,2}$ is adjacent to v, then e' retains in $G/_e v$ the same vertices and marking.

Case 2.2. One of $v_{1,2}$, say v_1, is adjacent to v. (Then v_2 is not adjacent to v by bipartacy.)

Case 2.2.1. If $v_1 = w$, then change v_1 to v in $G/_e v$, and retain the marking.

Case 2.2.2. So assume next $v_1 \neq w$. If v_2 is on the opposite side to e, then retain $v_{1,2}$ and the marking.

Case 2.2.3. If v_2 is on the same side as e, then we change v_1 to v, and retain the marking. (Note that by bipartacy, if v_2 is on the same side as e, then so must be v_1.)

In case of a signed (unmarked) edge e', the sign is copied (even if vertices are changed), except in the case 1.3, when e' receives a mark. In this case the sign of e' is deleted.

Since a mark will indicate for us only that the edge cannot be chosen as e, the resulting graph $G/_e v$ may be reduced by turning a multiple edge into a simple marked one. (This also makes it irrelevant to create a multiple edge in case 1.3.)

Definition 5.3 We can extend the definition 3.2 of $\operatorname{ind}(G)$ to marked graphs G by requiring in point 2 that e be simple *and unmarked*, and allowing at every stage the option that a multiple edge can be turned into a simple marked one. Then $\operatorname{ind}(G)$ coincides with the one previously defined if all edges of G are unmarked (in particular $\operatorname{ind}(D) = \operatorname{ind}(\Gamma(D))$ when all edges of $\Gamma(D)$ are understood unmarked).

If we further replace in definition 3.2 the two occurrences of $G_{v_i} = G/v_i$ by $G/_e v_i$, as given in definition 5.2, then we define the corresponding notions of 0-*independent* edges and the *modified index* $\operatorname{ind}_0(G)$. Again we set $\operatorname{ind}_0(D) = \operatorname{ind}_0(\Gamma(D))$ with all edges of $\Gamma(D)$ unmarked.

If one requires that in a set of 0-independent edges all be positive or negative, one obtains the modifications $\operatorname{ind}_{0,\pm}$ of $\operatorname{ind}_{\pm}$.

With this definition, we obtain (2). The property (3) can be proved by induction.

Lemma 5.1 For every marked graph G we have $\operatorname{ind}(G) \leq \operatorname{ind}_0(G)$.

Proof. We prove inductively over the vertex number of G that each independent set of edges in G is also 0-independent.

Let $e_1, \dots, e_n$ be independent in G. Thus e_1 is simple and unmarked in G, and $e_2, \dots, e_n$ are independent in G/v_1. Since G/v_1 is a contraction of $G/_e v_1$, by the proof of lemma 4.1, the edges $e_2, \dots, e_n$ are independent in $G/_e v_1$. By induction assumption they are thus 0-independent in $G/_e v_1$. Therefore, $e_1, \dots, e_n$ are 0-independent in G. □

This fixes Murasugi-Przytycki's proof of lemma 8.6 in [MP]. A similar fix works for (8) and (9).

We speculated whether in fact (2) can be stronger than (1). If (4) is false, then conjecture 3.1 is also false. We explained, though, that indeed (4) is true (and proved by Traczyk as a followup to our note). Still one needs our (much more awkward) definition of the modified index, at least for the scope of this note, in order to prove (3) or (4), and fix the gap in [MP].

Also, if one likes to keep the correspondence between (Seifert) graph and diagram, one must accept the circumstance that (in general) not all of $\operatorname{star} v$ is to be contracted. The idea of using vertex contraction (straightforwardly, following [MP]) appeared in at least one further paper, [MT]. To contain problems from propagating, we feel thus some priority justified to our point of caution. This motivated additionally the present correction.

6 Simple properties of the modified index

The important difference of ind_0 to ind lies in not affecting edges in case 1.2. The treatment of vertices on the opposite side to e, the technical detail missed by Murasugi-Przytycki, does not affect the result significantly, yet it creates a lot of calculation overhead (which we experienced in attempts to use the possibly better estimate (2) prior to Traczyk's proof of (4)). Note, however, that it implies the additivity of ind_0 under block sum in an easier (and much more natural) way than Murasugi-Przytycki's corresponding statement for ind.

Definition 6.1 A marked graph is *not* 2*-connected* if it has an *unmarked* edge whose deletion disconnects it. If G is not 2-connected, there is a plane curve intersecting G in a single, and unmarked, edge. We call such a curve a *separating curve*.

Note that the initial (unmarked Seifert) graph of D is 2-connected because we can assume D has no nugatory crossings.

Lemma 6.1 If G is 2-connected, so is $G/_e v$.

Proof. We assume to the contrary that $G/_e v$ is not 2-connected. Let e' be a disconnecting edge. So there is a separating curve γ that intersects $G/_e v$ only in e'. The only edges in $G/_e v$ which do not exist in G are of the type vz in (13). That is, z is a vertex on the same side as e, adjacent to a vertex x adjacent to v in G. (Note that when $x = w$, then all z adjacent to w are on the same side as e.) By definition 5.1, the property of z being on the same side as e implies that there is a cycle in G containing the edges $e = vw$, vx and xz. In $G/_e v$, this cycle is shortened when xz is replaced by vz (and e contracted). Thus vz belongs to a cycle in $G/_e v$, and cannot disconnect $G/_e v$.

Therefore, e' persists in G. It must be unmarked in G, since the move from G to $G/_e v$ never deletes marks. Thus the curve γ must intersect G in some other edge. The only edges added in G when recovering it from $G/_e v$ (except that e is decontracted) are of the form xz in (13) (with x a vertex adjacent to v, and z a vertex adjacent to x on the same side as e). Then γ passes in G through a cycle as the right one in (13) (the one containing z, x, v, w in consecutive order; note that $z \neq w$ by bipartacy). In $G/_e v$ this cycle is changed only by replacing xz and vx by vz (and contracting e). The only way γ can avoid this cycle in $G/_e v$ is that γ passes through $e' = vx$ in $G/_e v$ (and G). But by construction vx is marked in $G/_e v$, and γ is not a separating curve, a contradiction. $\square$

It is easy to see that $G_1 * G_2$ is 2-connected iff both G_1 and G_2 are so. This is true regardless of how (i.e., at which vertices) '$*$' is performed.

Lemma 6.2 If $G_{1,2}$ are 2-connected, then $\operatorname{ind}_0(G_1 * G_2) = \operatorname{ind}_0(G_1) + \operatorname{ind}_0(G_2)$.

Proof. It is enough to see that the contraction procedure of an edge e in G_1 does not affect edges or markings in G_2, except possibly the change of vertex at which the block sum $G_1 * G_2$ is performed.

Let v, w be the ends of e, and we consider the building of $G/_e v$ for $G = G_1 * G_2$. Let z be the (cut) vertex of G at which the block sum $G_1 * G_2$ is performed.

If $z \neq v$ is not adjacent to v, then nothing is changed in G_2 when building $G/_e v$.

Next assume $z = v$. The vertex v must be adjacent to at least one more vertex $x \neq w$ in G_1 (else G_1 is not 2-connected or e is multiple). Then we see with this choice of x in definition 5.1 that the vertices in G_2 except v lie on the opposite side to e. Thus building $G/_e v$ does not affect G_2.

Finally assume $z \neq v$, but z is adjacent to v. If $z = w$ is the other end of e, then in $G/_e v$ all edges incident in G_2 to w are redirected to v with the same marking, and so G_2 is not affected. If $z \neq w$, then choosing z for x in definition 5.1, we see that all vertices of G_2 except z are on the opposite side to e. Thus nothing of G_2 is affected by building $G/_e v$. $\square$

7 Outline of applications

We conclude by briefly outlining the context in which the above problem was encountered.

Beside the standard presentation of the braid groups using Artin's generators σ_i, another presentation has been studied for some time [BKL] by means of an extended set of *band generators* (and their inverses)

$$\sigma_{i,j}^{\pm1} = \sigma_i \dots \sigma_{j-2} \sigma_{j-1}^{\pm1} \sigma_{j-2}^{-1} \dots \sigma_i^{-1}$$

for $1 \le i < j \le n$. (Note that $\sigma_i = \sigma_{i,i+1}$.)

A representation of a braid β, and its closure link $L = \hat{\beta}$, as word in $\sigma_{i,j}^{\pm1}$ is called a *band representation*. A band representation of β spans naturally a Seifert surface of the link L: one glues disks into the strands, and connects them by half-twisted bands along the $\sigma_{i,j}$. The resulting surface is called *braided Seifert surface* of L.

A minimal genus Seifert surface of L occurring in the form of a braided Seifert surface is called a *Bennequin surface*. This term was coined by Birman-Menasco [BM] in honor of Bennequin, who had proved in [Be] that such surfaces exist for 3-braid links on a minimal (i.e., 3-strand) braid. It is known that not all links (or knots) carry a Bennequin surface on a minimal braid.

Murasugi-Przytycki's work (and its present correction) will be applied to obtain the following result:

Theorem 7.1 Any alternating knot of genus up to 4 or of at most 18 crossings

(a) makes the Morton-Williams-Franks inequality (6) sharp (i.e., an equality), except if it is the Murasugi-Przytycki knot or its mutant,

(b) satisfies conjecture 3.1 (for at least one alternating diagram D), and

(c) carries a Bennequin surface on a minimal braid.

The proof uses, among others, a computer implementation of Murasugi-Przytycki's graph algorithm. Details will be explained in a subsequent paper.

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