On non-conjugate braids with the same closure link

Alexander Stoimenow

Department of Mathematics, Keimyung University

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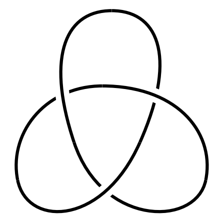
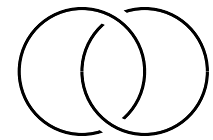
Braid closure and braid index

knot K $\qquad S^1 \hookrightarrow S^3$

link L $\qquad \underbrace{S^1 \cup \ldots \cup S^1}_{n \text{ components}} \hookrightarrow S^3$



K

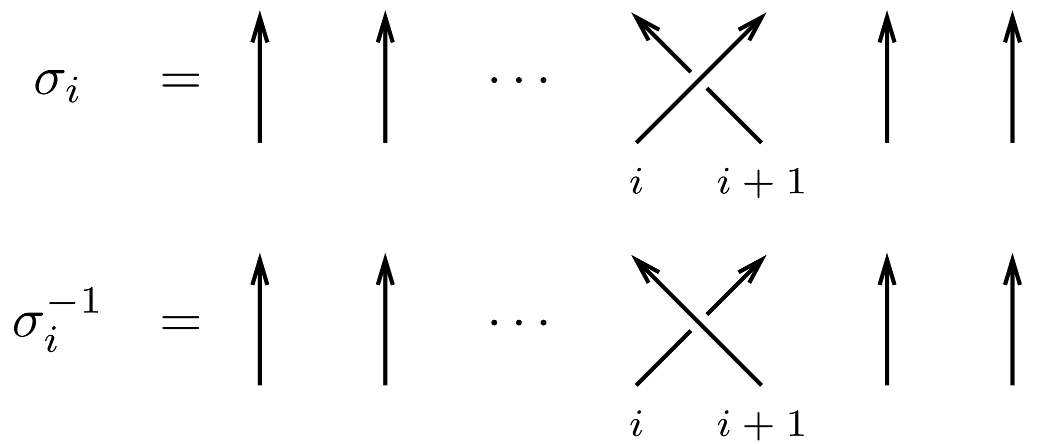


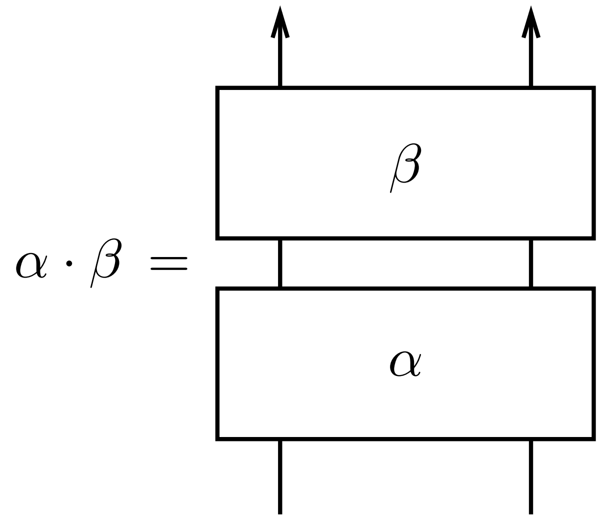
L

Definition 1 *The* braid group B_n *on* n *strands:*

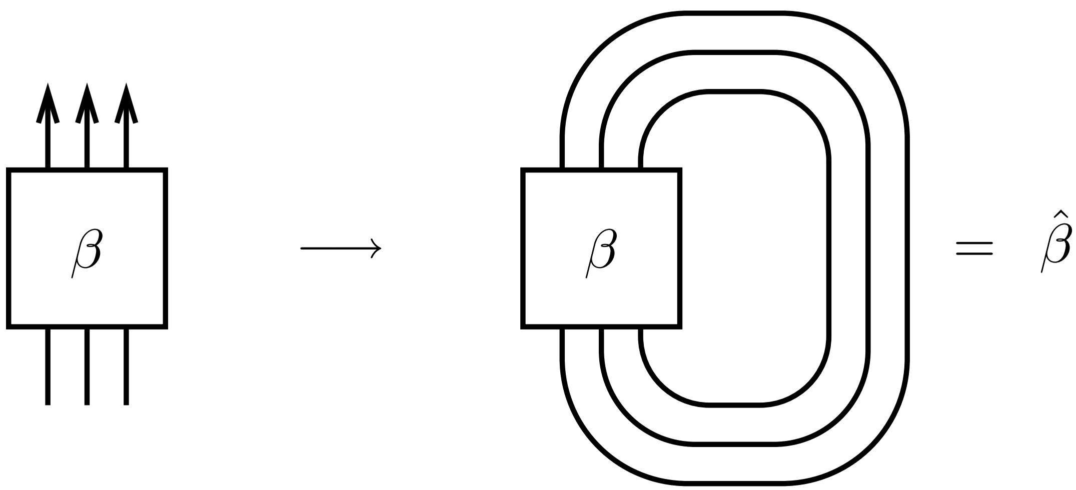
$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_j \sigma_i \sigma_j = \sigma_i \sigma_j \sigma_i & |i-j| = 1 \end{array} \right\rangle$$

σ_i – Artin standard generators. *An element* $\beta \in B_n$ *is an* n-braid.





Braid closure $\hat{\beta}$:



is a knot $S^1 \hookrightarrow S^3$ *or (more generally)* link $S^1 \cup \cdots \cup S^1 \hookrightarrow S^3$.

Note: # of components of $\hat{\beta}$ = # of cycles of *associated permutation* $\pi(\beta)$,

$$\pi \,:\, B_n \to S_n\,, \qquad \pi(\sigma_i) = \tau_i \;=\; (i\ i+1)\,.$$

We call β a *pure braid* if $\pi(\beta) = Id$.

Theorem 2 *(Alexander '23) Any link L is the closure of a braid β.*

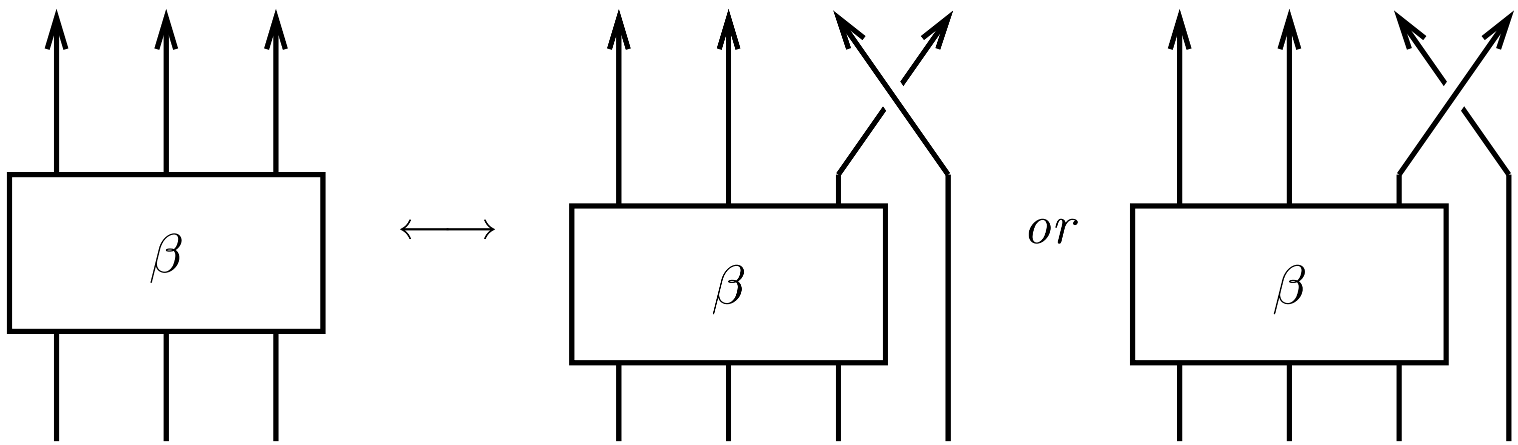
Definition 3 $b(L) \,:=\, \min \left\{ n \in \mathbb{N} \,:\, \exists \beta \in B_n,\, \hat{\beta} = L \right\}$ braid index *of* L.

We call β a *braid representation* of L. If $n = b(L)$ then β is called *minimal*.

Theorem 4 *(Markov '35, Birman '76) If $\hat{\beta}_1 = \hat{\beta}_2$, then $\beta_{1,2}$ are related by a sequence of*

1. *conjugacies in the braid group* $\beta \longmapsto \alpha\beta\alpha^{-1}$

2. *(de)stabilizations* $B_n \ni \beta \longleftrightarrow \beta\sigma_n^{\pm 1} \in B_{n+1}$



First move $\Rightarrow \{\, \beta \in B_n \,:\, \hat{\beta} = L \,\}$ is a union of conjugacy classes. Let

$$c(n, L) \;:=\; \# \text{ of conjugacy classes}\,.$$

Conjugacy is B_n is well-understood (Garside, $\ldots$) $\Rightarrow$ 'easy' part.
Second move (difficult part): how does it relate different conjugacy classes?

In this talk we will be mainly concerned with:

Question 5 *Fix a link L and $n \geq b(L)$. What is $c(n, L)$? In particular, when is $c(n, L) = \infty$?*

Exponent sum and Jones conjecture

How to distinguish conjugacy classes? Simplest invariant: exponent sum.

Definition 6 *$H_1(B_n) = B_n/[B_n, B_n] \simeq \mathbb{Z}$; homomorphism $[\,.\,]$ given by $\sigma_i \mapsto [\sigma_i] := 1$. $[\beta]$ is* writhe *or* exponent sum *of β.*

For a link L consider

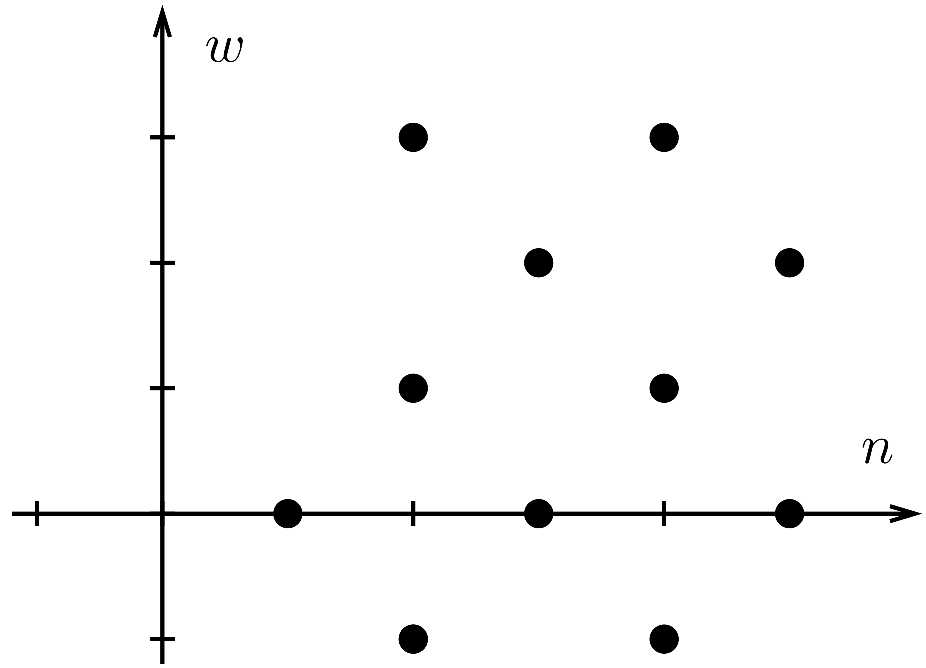
$$W_L := \{\, (n, w) \in \mathbb{N} \times \mathbb{Z} \,:\, \exists\, \beta \in B_n \,:\, [\beta] = w\,,\ \hat{\beta} = L \,\}$$

and

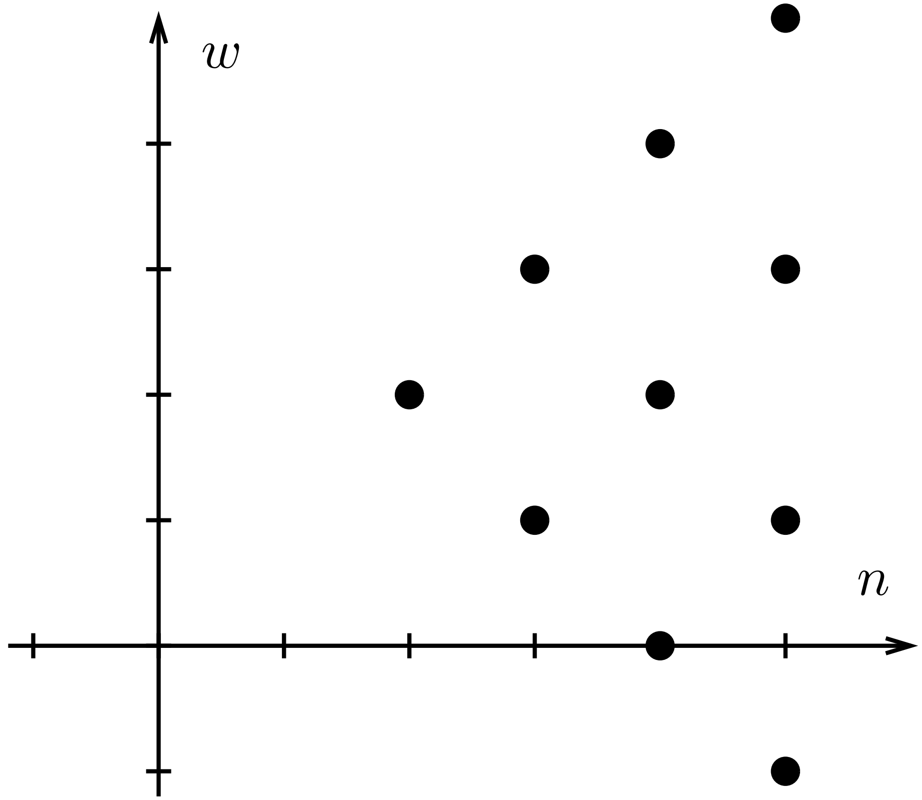
$$w(n, L) := \#\{\, w \,:\, (n, w) \in W_L \,\}$$

(the number of exponent sums of n-braid representations of L).

For example, if L knot $\Longrightarrow$ $n+w$ odd, so $W_L \subset$ right diagram.



Stabilization shows that when $(n, w) \in W_L$, then W_L contains the "cone" $C(n, w)$ of (n, w), made up of all $(n+k, w+m)$, for $k > 0$, $|m| \leq k$ with $m+k$ even:



Corollary 7

$$w(n, L) \geq n - b(L) + 1\,. \tag{1}$$

Corollary 8 $c(n, L) \geq n - b(L) + 1\,.$

Exponent sum distinguishes *some* conjugacy classes when $n > b(L)$.

Conjecture 9 *(Jones '87)*
1. (weaker version) (1) *is exact for* $n = b(L)$, *i.e.* $W_L \cap (\{b(L)\} \times \mathbb{Z})$ *is one point* $=: (b(L), w(L))$ *('Exponent sum of minimal representations is unique')*
2. (stronger version) (1) *is exact for all* $n \geq b(L)$, *i.e.* $W_L = C(b(L), w(L))$.

It is known for certain classes of links. For example, Birman-Menasco proved:

Theorem 10 *(Birman-Menasco '93) Up to 3 strands only one minimal conjugacy class, and the conjugacy classes of its stabilizations, have same closure link.*

$\Rightarrow$ at most 3 classes have same closure link.
$\Rightarrow$ Jones conjecture is true when $3 \geq n \geq b(L)$. (I later showed $(n >)3 \geq b(L)$.)

One main tool to prove Jones conjecture: *Morton-Williams-Franks inequality*. (We will not discuss it further in detail here.) It shows:

Corollary 11 *For every link* L *and any* $n \geq b(L)$, *we have* $w(n, L) < \infty$.

So the exponent sum is (not surprisingly) too weak to distinguish infinitely many conjugacy classes of n-braid representations.

Constructing infinitely many non-conjugate braids

The first result is due to Morton:

Theorem 12 *(Morton '78) There is an infinite sequence of conjugacy classes of 4-braids β with closure $\hat{\beta} = \bigcirc$ (unknot).*

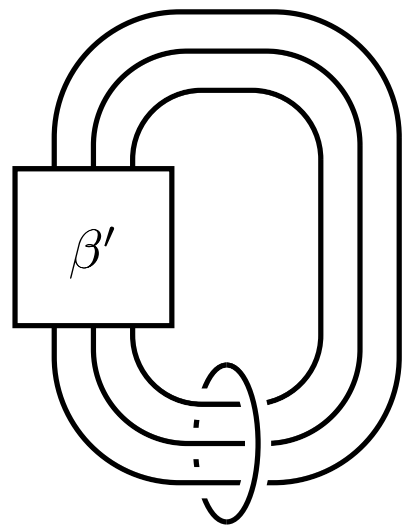
More recently: $(2, k)$-torus links (Fukunaga '04). Then knots (Shinjo):

Theorem 13 *(Shinjo '05)* K knot, $n \geq 4$, $\exists$ $n-1$*-strand braid* β *with closure* $\hat{\beta} = K \Rightarrow \exists$ *infinitely many non-conjugate* β' *of* n *strands with* $\hat{\beta}' = K$.

Proof. Conjugate β and stabilize

$$\beta' = \alpha\beta\alpha^{-1}\sigma_n \,. \tag{2}$$

Consider *axis addition link* of β':



Conjugate braids have the same axis addition link. If one can distinguish axis addition links by some link invariant, then braids are not conjugate. Use (the second lowest term of) the *Conway polynomial* ∇ (and choose α well). $\square$

First step of extension:

Theorem 14 *(S. '06) In theorem 13, K does not need to be a knot, but β should be a non-pure braid. Then $c(n, K) = \infty$.*

Proof. Mainly refining Shinjo's argument. $\square$

How about pure braids? Shinjo's argument fails.

Definition 15 *If $\beta \in B_n$ is pure, one can define the i-th strand for $1 \leq i \leq n$ on the conjugacy class of β. A* subbraid *of β is obtained by taking the i-th strands for $i \in I$ and $I \subset \{1, \dots, n\}$.*

Fact: The *center* of B_n (elements that commute with all B_n) is infinite cyclic and generated by $(\sigma_1 \dots \sigma_{n-1})^n$.

Theorem 16 *(S. '06) If $\beta \in B_{n-1}$ with $L = \hat{\beta}$ and $n \geq 4$ is pure and β_0 is not central for some 3-strand subbraid β_0 of β, then $c(n, L) = \infty$.*

Proof. We use the *Burau representation.*

The Burau representation ψ of B_3 is a homomorphism into algebra of 2×2 matrices over $\mathbb{Z}[t, t^{-1}]$:

$$\psi(\sigma_1) = \begin{bmatrix} -t & 0 \\ -1 & 1 \end{bmatrix}, \quad \psi(\sigma_2) = \begin{bmatrix} 1 & -t \\ 0 & -t \end{bmatrix}.$$

The *Burau trace* $\operatorname{tr}\psi$ is a conjugacy invariant. Some linear algebra: construction (2) works unless $\psi(\beta_0)$ is a scalar matrix (all off-diagonal entries 0, all diagonal entries same). The Burau representation is faithful in the case of B_3 $\Rightarrow \beta_0$ is central. $\square$

Application of Lie group theory

Theorem 17 *(S. '08) If $\beta \in B_{n-1}$ with $\hat{\beta} = L$ is not central, $n \geq 4$, then $c(n, L) = \infty$.*

Proof. Entirely different (global) approach. Now we let α in (2) vary over all of B_{n-1} instead of constructing them explicitly.

We use the n-strand Burau representation $\psi_n : B_n \to GL(n-1, \mathbb{Z}[t, t^{-1}])$.

Theorem 18 *(Squier '84) When $t \in \mathbb{C}$ and $|t| = 1$, then ψ_n preserves a Hermitian form on $\mathbb{C}^{n-1}$. This form is definite for $t \approx 1$, so $Im\ \psi_n \subset U(n-1)$.*

Theorem 19 *(Freedman-Larsen-Wang '02, Marin '07, S.-Yoshino '06) For 'proper' $t \approx 1$, $Im\ \psi_n$ is dense in $U(n-1)$.*

(Use Dynkin's classification of maximal subgroups of the classical Lie groups.)

$\operatorname{tr} \psi_n(\beta')$ gives a linear condition on $\psi_{n-1}(\alpha\beta\alpha^{-1})$, which is not a trace.

Proposition 20 *A non-trace linear condition is not satisfied on any non-central conjugacy class in $SU(n)$.*

$\ldots \Rightarrow$ done unless $\psi_{n-1}(\beta)$ is central $\overset{?}{\Longrightarrow}$ β is central ??

But: the Burau representation is *not* faithful in the case $n \geq 5$ (Bigelow, Long-Paton, Moody; for $n = 4$ open)!

We use the faithful *Lawrence-Krammer representation* and repeat the whole proof (incl. theorem 19, etc., which is now more difficult). $\square$

Extending Shinjo's construction and exchange moves

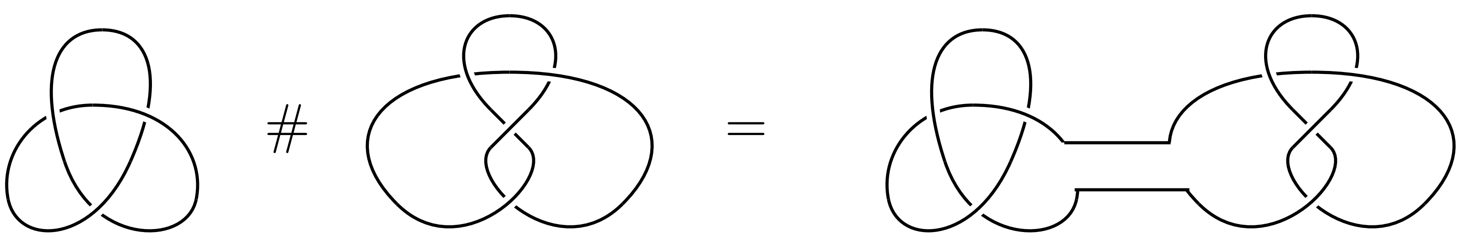
The case $n = b(L)$ of minimal conjugacy classes is *far* more difficult:

- Birman '69 conjectured $c(b(L), L) = 1$ for all L. Murasugi-Thomas '72 disproved this.
- But true for $L =$ unlink (remark in problem 28) and $\exists$ further example (Ko-Lee '98).
- *Apparently* (no clear reference) true for $L =$ torus link.
- Let
$$\Delta_n^2 = (\sigma_1 \cdot \ldots \cdot \sigma_{n-1})^n$$
be the full twist on n strands. Tetsuya Ito (in a recent email to Shinjo) *announced*:
  (a) If $\beta \in B_n$ has the form $\beta = \Delta_n^2 \alpha$ with α *positive* (i.e. no σ_i^{-1}), then $(b(\hat{\beta}) = n$ and) $c(n, \hat{\beta}) < \infty$.

(b) If $\beta = \Delta_n^{2n}\alpha$ with α positive, then $c(n, \hat{\beta}) = 1$.

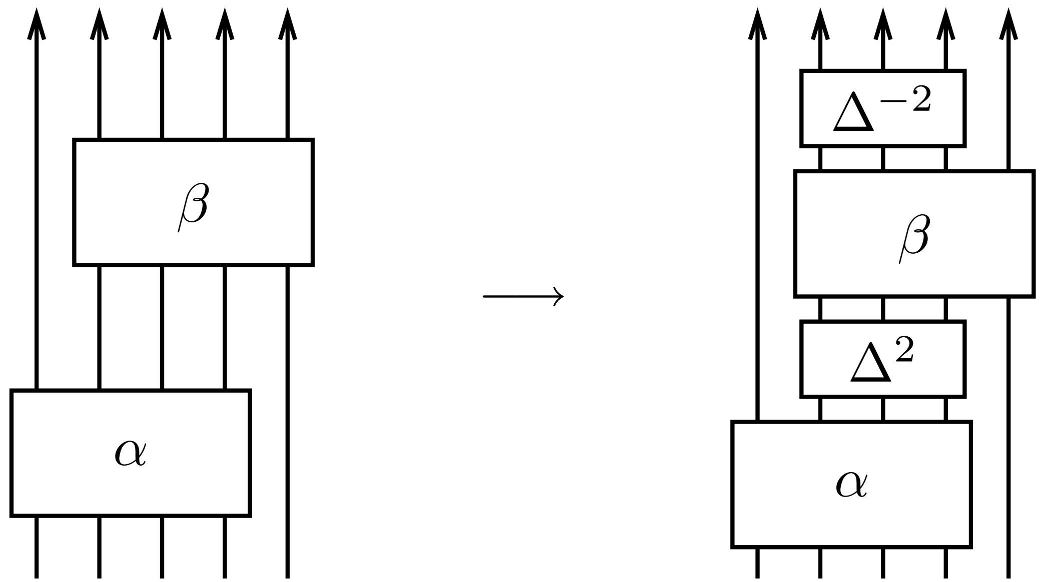
So *some* links have $c(b(L), L) < \infty$. But many links have $c(b(L), L) = \infty$. E.g.

Proposition 21 *(S. '06)* L composite *link* $L_1 \# L_2$ *with* $b(L) \geq 4$ *such that the components of* $L_{1,2}$ *on which* $\#$ *is performed are knotted. In particular any composite knot:*



Then $c(b(L), L) = \infty$.

Birman-Menasco '92 introduced the *exchange move*. In its most general form, it looks like:



, (3)

where $\Delta^2 = \Delta^2_{n-2}$ is the full twist on $n-2$ strands.

It underlies the switch between conjugacy classes with the same closure link, in a universal way.

Theorem 22 *(Birman-Menasco '92) The n-braid representations of a given link decompose into a finite number of classes under the combination of exchange moves and conjugacy.*

Let us say that an n-braid *admits an exchange move* if it has a word as on the left of (3).

Corollary 23 *If $c(n, L) = \infty$, then L has an n-braid representation admitting an exchange move.*

In a joint project with Reiko Shinjo we are about to prove 'most' of the *converse*:

Theorem 24 *(Shinjo-S. '10; <u>almost</u>) Let L have an n-braid representation β admitting an exchange move with $\pi(\beta)(1) \neq 1$ and $\pi(\beta)(n) \neq n$. Then $c(n, L) = \infty$.*

Corollary 25 *Let K be a knot, or a link without trivial components, and let $n \geq b(K)$. Then $c(n, K) = \infty$ iff K has an n-braid representation admitting an exchange move.*

$\Longrightarrow$ Proposition 21 for *knots*

Example 26 *All prime knots K of crossing number $c(K) \leq 10$ and $b(K) \geq 4$, except 7 knots for $c(K) = 9$ and 15 knots for $c(K) = 10$ have a minimal representation admitting an exchange move $\Rightarrow c(b(K), K) = \infty$.*

Remark 27 (regarding Theorem 24)

- Note that the exchange move is trivial when the leftmost strand of α or the rightmost strand of β are isolated. So the condition on $\pi(\beta)$ is needed. (It is the weakest condition in terms of $\pi(\beta)$ alone under which the exachange move can work.)

- Using a construction of Stanford, one can 'approximate' these cases of failure by others which cannot be told apart by any number of Vassiliev invariants (including coefficients of ∇).

- It appears hardly realizable to extend the Lie group approach to any useful degree for exchange moves. This is because the choice of α in (2) is now very restricted (to powers of Δ^2).

Open questions

Theorem 17 is the maximum one can get out of (2). It settles question 5 almost fully for $n > b(L)$. The most interesting missing case is:

Problem 28 *Let L be the trivial $n-1$-component link* $\underbrace{\bigcirc\bigcirc\cdots\bigcirc}_{n-1}$ (unlink*),*
$n \geq 4$. Is a braid representation $\beta' \in B_n$ of L conjugate to $\sigma_1^{\pm 1}$? (Remark: Birman-Menasco $\Rightarrow$ $\beta \in B_{n-1}$ is trivial.)

Problem 29 *The case $n = b(L)$ of minimal conjugacy classes is still difficult. Theorem 24 suggests to seek braids admitting exchange moves, but how can we identify what links have such (minimal) braids?*

Problem 30 *We don't know about β with $n > b(L)$ which are not conjugate to stabilizations (*Markov irreducible*): known only for $n = 4$ and $L =$ unknot (Morton, Fiedler).*

Thank you!

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