Introduction to Topology

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Chapter 1. Set Theory and Logic

1. Fundamental Concepts

$$\text{set} = \{\text{objects}\}$$

some logical fundamentals of set theory needed.

$P = \{A : A \notin A\}$, then $P \notin P \Rightarrow P \in P \Rightarrow P \notin P$

Russell's paradox! but not discussed here.

Basic Notation

$a \in A$	element
$a \notin A$	not element
$A \ni a$	A contains a
$A \subset B$	every $a \in A$ has $a \in B$
	includes = $A \subset A$
	if excludes = write $A \subsetneq B$ (proper inclusion)
$B \supset A$	B superset of A or B contains A

notation of sets

{objects}	$\{1, 2, 3\}$
{objects : property}	$\{x : x$ even int.$\}$

union/intersection/or/and

$A \cup B$	union	$A \cup B$	$= \{x : x \in A \text{ or } x \in B\}$
$A \cap B$	intersection	$A \cap B$	$= \{x : x \in A \text{ and } x \in B\}$
			$= \{x \in A : x \in B\}$

Remark or is not exclusive
"P or Q" means ok if P and Q
if we want to exclude P and Q
say "either P or Q" or "P or Q but not both"

$\varnothing$ empty set
if $A \cap B = \varnothing$ then A, B disjoint
$\forall x, x \notin \varnothing$
$A \cup \varnothing = A, A \cap \varnothing = \varnothing$

if ... then

If P, then Q $\quad P \Rightarrow Q$ means if P = true, then Q = true.
Q if P $\quad$ if P = false, then Q = true or false.
P if and only if Q $\quad P \Leftrightarrow Q$

Ex. "if $x^2 < 0$, then $x = 23$"
$P(x) =$ "$x^2 < 0$". Since $\forall x\,(P =$ false), "$x = 23$" (vacuously) true.

contrapositive/converse

$P \Rightarrow Q$ (contrapositive) $\neg Q \Rightarrow \neg P$
Contrapositive is true if and only if the statement is true.
Contrapositive is logically equivalent.

$P \Rightarrow Q$ (converse) $Q \Rightarrow P$ (not equivalent to $P \Rightarrow Q$)

$P \Leftrightarrow Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$

Negation of quantified statements

(negation)
$\forall x \quad P \quad$ "$\neg(\forall x \quad P)$" = "$\exists x \quad \neg P$"
$\exists x \quad P \quad$ "$\neg(\exists x \quad P)$" = "$\forall x \quad \neg P$"

Set difference

$A - B = A \setminus B = \{x \in A : x \notin B\}$

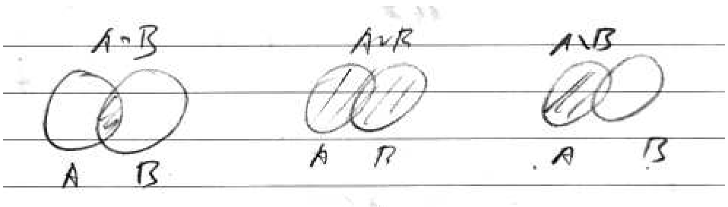

Figure 1- 1. Venn diagrams

$A \cap B = B \cap A \qquad A \setminus B \neq B \setminus A$

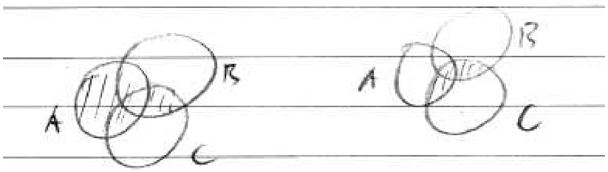

Figure 1- 2. Venn diagrams

$A \cup (B \cap C) \neq (A \cup B) \cap C$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ distributive law
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ DeMorgan's law
$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Collection of Sets

can form sets out of sets

$$\mathcal{P}(\mathcal{A}) := \{\mathcal{B} : \mathcal{B} \subset \mathcal{A}\} \quad \text{power set of A}$$

$$a \in A \Leftrightarrow \{a\} \subset A \Leftrightarrow \{a\} \in \mathcal{P}(\mathcal{A})$$

Arbitrary Unions and Intersections

$\mathcal{A}$ family of sets

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A = \{x : \exists A \in \mathcal{A} \quad x \in A\}$$

$$x \text{ is in at least one } A \in \mathcal{A}$$

$$\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} A = \{x : \forall A \in \mathcal{A} \quad x \in A\}$$

$$x \text{ is in all } A \in \mathcal{A}$$

Cartesian Products

$$A \times B = \{(a,b) =: a \times b \quad : \quad a \in A, b \in B\}$$

2. Functions

We need a bit more formal definition to make precise domain

C, D sets, A rule of assignment is $r \subset C \times D$ with

$$\forall c \in C \qquad (\{c\} \times D) \cap r \text{ has at most one element}$$

Then define

$$domain(r) = \{c \in C : (\{c\} \times D) \cap r \neq \varnothing\}$$
$$image(r) = \{d \in D : (C \times \{d\}) \cap r \neq \varnothing\}$$

Definition $f = (r, B)$ is a function — r is rule of assingment, $B \supset image(r)$
$A := dom(r) = dom(f)$ — domain 정의역
공역 range of f — $B \supset image(r) = image(f)$ — image of f 치역

[**Remark** elsewhere (my other lectures) image = range 치역
analysis range=target 공역]

$f = (r, B) : A \to B \quad a \in B$ let $b \in B$ be so that $(a, b) \in r$
(b unique!) $f(a) = b$ — image of a under f
value of f at a

Definition $g : A \to B \qquad A_0 \subset A$, where $g = (r, B), r \subset C \times D$
$g : A_0 \to B$ defined by $(r \cap A_0 \times D, B)$
$g|A_0$ restriction of g to A_0

Ex. $f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x^2$
$\overline{\mathbb{R}_+} = [0, \infty)$ $\quad g : \overline{\mathbb{R}_+} \to \mathbb{R} \qquad g(x) = x^2$
$h : \mathbb{R} \to \overline{\mathbb{R}_+} \qquad h(x) = x^2$
$k : \overline{\mathbb{R}_+} \to \overline{\mathbb{R}_+} \qquad k(x) = x^2$

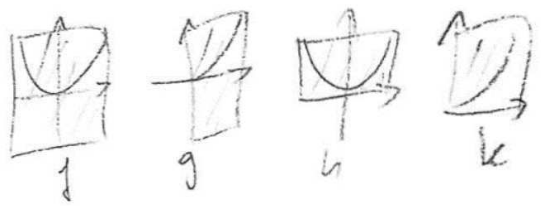

Figure 1- 3. Function f, g, h, k

(for us) are all diff, $g = f|_{\overline{\mathbb{R}_+}}, k = h|_{\overline{\mathbb{R}_+}}$
(we keep record of range!)

Definition $f : A \to B \qquad g : B \to C$
Define composite $g \circ f : A \to C \qquad g \circ f(a) = g(f(a))$
rule of assignment $r_{g \circ f} = \{(a, c) \in A \times C : \exists b \in B \ (a, b) \in r_f \ (b, c) \in r_g\}$

Note we define $g \circ f$ only if $range(f) \underset{!}{=} dom(g)$ (not subset)

Definition

$f : A \to B$	injective (one-to-one)	if $\forall a \neq b \quad a, b \in A \quad f(a) \neq f(b)$ ($\Leftrightarrow f(a) = f(b) \Rightarrow a = b$)
$f : A \to B$	surjective (onto)	if $\forall b \in B \quad \exists a \in A \quad f(a) = b$ ($\Leftrightarrow image(f) = range(f) = B$)
$f : A \to B$	bijective (1-to-1 correspondence)	if f injective & surjective

If $f : A \to B$ bijective, then exists $f^{-1} : B \to A$ inverse

$b \in B \quad f^{-1}(b) = a$ with $f(a) = b$ (f bijective $\to \exists! a$)

$f : A \to A \quad f(x) = x \quad Id_A$ identity function

lemma $f : A \to B$

if $\exists g, h : B \to A \quad g \circ f = Id_A$ left inverse

$f \circ h = Id_B$ right inverse

Then f bijective and $g = h = f^{-1}$

Definition $f : A \to B \qquad A_0 \subset A$

$$f(A_0) = \{f(a) : a \in A_0\} \quad \text{image of } A_0 \text{ under } f$$

$$= \{b \in B : \exists a \in A_0 \; f(a) = b\}$$

$B_0 \subset B \qquad f^{-1}(B_0) = \{a \in A : f(a) \in B_0\}$ preimage of B_0 under f

Remark If f bijective, then $\underset{\text{preimage}}{f^{-1}(B_0)} = \underset{\text{image under inverse map}}{(f^{-1})(B_0)}$

Remark $f(f^{-1}(B_0)) \subset B_0$ " = "if f surjective

$f^{-1}(f(A_0)) \supset A_0$ " = "if f injective

3. Relations

(1) equivalence relation (2) order relation

Definition A relation on a set A is a subset $C \subset A \times A$

xCy if $(x, y) \in C$ "x is in the relation C to y"

$x \sim_C y$

$x \nsim_C y \quad (x, y) \notin C$ "x is not in the relation C to y"

(1) equivalence relation

Definition $C \subset A \times A$ equivalence relation if

(1) Reflexivity $\quad x \sim_C x \quad \forall x \in A$

(2) Symmetry $\quad \forall x, y \in A \quad x \sim_C y \Rightarrow y \sim_C x$

(3) Transitivity $\quad \forall x, y, z \in A \quad x \sim_C y, y \sim_C z \Rightarrow x \sim_C z$

Ex. $C = A \times A$ trivial equivalences

$C = \{(x, x) : x \in A\}$

$A = \mathbb{Z} \quad x \sim y \Leftrightarrow y - x$ is even

Definition $C \subset A \times A$ equivalence relation

$a \in A, a \in [a] \qquad [a] = [a]_{\sim_C} = \{b \in A : b \sim_C a\} \subset A$

lemma $\forall a, b \in A \qquad [a] = [b]$ or $[a] \cap [b] = \varnothing$

Two equivalence classes are equal or disjoint

Definition A set, $\mathcal{E}$ family of sets is partition of A if

$a)$ $\forall A' \in \mathcal{E} \quad \varnothing \neq A' \subset A$ nonempty subsets of A

$b)$ $\forall A', A'' \in \mathcal{E} \quad$ if$A' \neq A'' \Rightarrow A' \cap A'' = \varnothing$ disjoint

$c)$ $\bigcup_{A' \in \mathcal{E}} A' = A$

$\{\text{equivalence relations on } A\} \Leftrightarrow \{\text{partitions of } A\}$

If $\sim_C$ is equivalence relation of A $\Rightarrow \mathcal{E} =\{\text{equivalence classes}\}$ is a partition of A

If $\mathcal{E}$ partition of A, define $\forall a, b \in A$ $a \sim_{\mathcal{E}} b$ if $\exists A' \in \mathcal{E}$ $a, b \in A'$ is an equivalence relation

2 order relations (more important for us)

Definition $C \subset A \times A$ (simple) order, linear order if
(1) Comparability $\forall x, y \in A$ $x \neq y$ $x \sim_C y$ or $y \sim_C x$
(2) Nonreflexivity $\forall x \in A$ $x \nsim_C x$
(3) Transitivity $\forall x, y, z \in A$ $x \sim_C y, y \sim_C z \Rightarrow x \sim_C z$

Remark $(2), (3) \Rightarrow$ "or" in (1) is either or
($\nexists$ $x, y \in A : x \sim_C y$ and $y \sim_C x$)

Ex. $A = \mathbb{R}$ $<$ usual order relation
$A = \mathbb{R}$ $x \sim_C y \Leftrightarrow x^2 < y^2$ or $x^2 = y^2$ $x < y$
unusual order relation

Ex. (A, C) C order relation $B \subset A$ $(B, C \cap (B \times B))$ restriction

usually use '$<$' for order relation

Definition $x < y$ x less than y, $(A, <)$ ordered set
$x \leq y \Leftrightarrow x < y$ or $x = y$
$x > y \Leftrightarrow x < y$
$y \geq x \Leftrightarrow y > x$ or $y = x$
write $x < y < z$ for $x < y$ and $y < z$

Definition $a, b \in (A, <)$ $a < b$
Define $(a, b) = \{x : a < x < b\}$ open interval
if $(a, b) = \varnothing$ call b immediate successor of a $b = \text{succ}(a)$
a immediate predecessor of b $a = \text{pred}(b)$

Ex. $(\mathbb{R}, <)$ no element has immediate succesor and predecessor
because $\forall a < b$ $(a, b) \neq \varnothing$
$(\mathbb{Q}, <)$ similar
$(\mathbb{Z}, <)$ immediate succesor and predecessor of $a \in \mathbb{R}$ is $a + 1$ and $a - 1$, respectively

Definition $(A, <_A) \simeq (B, <_B)$ same order type if
$\exists f : A \to B$ bijective and $a_1 <_A a_2 \Leftrightarrow f(a_1) <_B f(a_2)$

Ex. $\mathbb{R} \simeq (-1, 1)$ $f(x) = \dfrac{x}{1 - x^2}$

$[0, 1) \simeq \{0\} \cup (1, 2)$ $f(0) = 0$ $f(x) = x + 1$

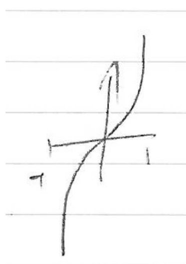

Figure 1 - 4: $x/(1 - x^2)$

Definition $(A, <_A), (B, <_B)$ ordered sets
Define $<_{A\times B}$ on $A \times B$ by
$(a_1, b_1) < (a_2, b_2)$if $a_1 <_A a_2$ or $a_1 = a_2$ and $b_1 <_B b_2$
It is called dictionary(lexicographical) order relation

Ex. $\mathbb{Z} \times [0,1)$ with usual order has the order type of $\mathbb{R}$ with lexicographical order
$f(z,\epsilon) = z + \epsilon$

$[0,1) \times \mathbb{Z}$ has very different order type
every element has an immediate predecessor and successor

Bounds, Maxima, Suprema, ...

$(A, <)$ ordered set, $A_0 \subset A$

$b = \max A_0$ largest element of A_0 if $b \geq x \quad \forall x \in A_0$
$b = \min A_0$ smallest element of A_0 if $b \leq x \quad \forall x \in A_0$

$b \in A$ upper bound for A_0 if $b \geq x \quad \forall x \in A_0$
$b \in A$ lower bound for A_0 if $b \leq x \quad \forall x \in A_0$ (does not need to exist!)

A_0 bounded above if A_0 has upper bound
A_0 bounded below if A_0 has lower bound
bounded = bounded above + bounded below

A_0 has least upper bound b if
b upper bound for A_0 and $\forall x \in A$ $x < b$: x is not upper bound for A_0

$$b = \min\{x : x \text{ upper bound for } A_0\}$$
$$b = \sup A_0 \quad \text{supremum}$$

A_0 has greatest lower bound b if
b lower bound for A_0 and $\forall x \in A$ $x > b$ x is not lower bound for A_0

$$b = \max\{x : x \text{ lower bound for } A_0\}$$
$$b = \inf A_0 \quad \text{infimum}$$

Remark if $b = \sup A_0 \in A_0$, then $b = \max A_0$
if $b = \inf A_0 \in A_0$, $b = \min A_0$

Definition $(A, <)$ has least upper bound(l.u.b.) property
if $\forall A_0 \subset A \quad A_0 \neq \varnothing \quad A_0$ bounded above
$\exists \sup A_0 \in A$

$(A, <)$ has greatest lower bound(g.l.b.) property
if $\forall A_0 \subset A \quad A_0 \neq \varnothing \quad A_0$ bounded below
$\exists \inf A_0 \in A$

Theorem $(A, <)$ has g.l.b.p. $\Leftrightarrow$ l.u.b.p.

Ex. Assume $(\mathbb{R}, <)$ has l.u.b.p. (discuss this later!)
then $A = (-1, 1), <$ has l.u.b.p.

proof. $A_0 \subset (-1, 1)$ $A_0 \neq \varnothing$ bounded above in A_0
$\exists b \in (-1, 1)$ $b \geq x$ $\forall x \in A_0$
$b \in A_0 \subset \mathbb{R} \Rightarrow A_0$ bounded above in $\mathbb{R}$
$\mathbb{R}$ has l.u.b.p. $\Rightarrow \exists$ least upper bound $\tilde{b}$ of A_0 in $\mathbb{R}$
$A_0 \neq \varnothing$ $\exists a \in A_0$ $a \leq \tilde{b} \leq b$
now $a, b \in A_0 = (-1, 1) \Rightarrow \tilde{b} \in (-1, 1) = A$
$\tilde{b}$ is least upper bound of A_0 in A
Similarly all intervals in $\mathbb{R}$ have l.u.b.p.

Ex. $A = (-1, 0) \cup (0, 1)$
$A_0 = \{-1/n : n > 1\}$ has upper bound but no least upper bound

4. Integers and Real numbers

need a bit more formal approach to real numbers via axioms

Definition $f : A \times A \to A$ binary operation on A
$f(a, a') = afa'$

define group, Abelian group, field

Definition The real numbers $(\mathbb{R}, +, \cdot, <)$ is a set with two binary operations
$+$ addition, $\cdot$ multiplication
and one ordering relation $<$ such that

(1)-(5) $(\mathbb{R}, +, \cdot)$ is a field

Mixed algebraic and order property
(6) $\forall x, y, z \in \mathbb{R}$ $x > y \Rightarrow x + z > y + z$
$x > y, z > 0 \Rightarrow x \cdot z > y \cdot z$

order properties
(7) $<$ has least upper bound property
(8) if $x < z$ $\exists y : x < y$ and $y < z$

$-x$ is the additive inverse $x + (-x) = 0$
$a - b = a + (-b)$ subtraction

$x \neq 0$ $\frac{1}{x} = x^{-1}$ is multiplicative inverse $x\,\frac{1}{x} = 1$

$b \neq 0$ $\frac{a}{b} = a \cdot b^{-1} = b^{-1} \cdot a$ quotient

all other common properties of real numbers can be derived from these axioms (1)-(8)

Ex. if $x > y$ $z < 0 \Rightarrow x \cdot z < y \cdot z$
$-1 < 0 < 1$

$$\left.\begin{array}{l}(1)-(5)\text{ field}\\(6)\end{array}\right\}\ \text{ordered field } (\Rightarrow \text{char} = 0!)$$
$$(7),(8) \quad \}\ \text{linear continuum (topological term)}$$

Remark $(8) \Leftarrow (1)-(7)$
given $x \neq z$ build $y = \dfrac{x+z}{2}$ with $2 = 1+1 \neq 0(!!)$

Definition $x > 0$ positive
$x < 0$ negative

formal definition of integers

$$A \subset \mathbb{R} \text{ inductive if } 1 \in A \text{ and } \forall x \in A\ x+1 \in A$$
$$\mathcal{A} = \{A \in \mathbb{R} : A \text{ inductive}\}$$
$$(\mathbb{N}_+ =)\mathbb{Z}_+ = \bigcap_{A\in\mathcal{A}} A \text{ positive integers (natural numbers } \mathbb{N})$$
$$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \underbrace{-\mathbb{Z}_+}_{\{-a:a\in\mathbb{Z}_+\}}$$

Remark $\mathbb{Z}_+ \subset \mathbb{R}_+ = (0,\infty)$ because $(0,\infty)$ inductive
$\min\mathbb{Z}_+ = 1.....[1,\infty)$ is inductive, so $\mathbb{Z}_+ \subset [1,\infty)$

define rational numbers

$$\mathbb{Q} = \{m/n : m,n \in \mathbb{Z}, n \neq 0\}$$

Theorem 4.1 (Well-ordering property)
$A \subset \mathbb{Z}_+\ A \neq \varnothing \Rightarrow \exists \min A\ A \ni$ smallest element

$\{1,\cdots,n\} = S_{n+1}$ section of positive integers $S_1 = \varnothing$

Theorem 4.2 (Strong induction principle)
$A \subset \mathbb{Z}_+$ and $\forall n \in \mathbb{Z}_+\ S_n \subset A \Rightarrow S_{n+1} \subset A$
(in particular, $n = 1\ \varnothing = S_n \subset A \Rightarrow S_2 = \{1\} \subset A \Rightarrow 1 \in A$)
then $A = \mathbb{Z}_+$

l.u.b. axiom (7) $\Rightarrow$ $\mathbb{Z}_+$ has no upper bound
(Archimedean ordering property)
$\Rightarrow$ $\exists\sqrt{x}\ x > 0\cdots\sqrt{a} = \sup\{x : x\cdot x \leq a\}$

5. Cartesian products

Generalize $A \times B$

Definition $\mathcal{A}$ family of sets
indexing function $f : J \longrightarrow \mathcal{A}$ f surjective J index set
$\alpha \in J$ write $f(\alpha) \in \mathcal{A}$ as A_α
$\mathcal{A} = \{A_\alpha\}_{\alpha\in J}$
We don't need f bijective so some set can be indexed multiply!

$$\bigcup_{\alpha\in J} A_\alpha = \bigcup_{A\in\mathcal{A}} A = \{x : A_\alpha \ni x \text{ for at least one } \alpha \in J\}$$
$$\bigcap_{\alpha\in J} A_\alpha = \bigcap_{A\in\mathcal{A}} A = \{x : A_\alpha \ni x \text{ for all } \alpha \in J\}$$

Ex. $J = \{1, 2\}$ $\mathcal{A} = \{A_1, A_2\}$
$\bigcap\limits_{\alpha\in J} A_\alpha = A_1 \cap A_2$ $\bigcup\limits_{\alpha\in J} A_\alpha = A_1 \cup A_2$

Definition $m \in \mathbb{Z}_+ = \mathbb{N}$ X set
m-tuple x of elements in X is $x : \{1, \dots, m\} \to X$
$x = (x(1), \dots, x(m))$ $x(i) \in X$ i-th coordinate of x

$$\mathcal{A} = \{A_1, \dots, A_m\} \text{ family indexed by } \{1,\dots,m\}$$

$$A_1 \times \cdots \times A_m = \{x : m\text{-tuple of } X = \bigcup_{i=1}^{m} A_i \text{ with } x(i) \in A_i \forall i = 1, \dots, m\}$$

Remark $(A \times B) \times C \simeq A \times (B \times C) \simeq A \times B \times C$
$((a, b), c) \leftrightarrow (a, (b, c)) \leftrightarrow (a, b, c)$

$$n \in \mathbb{Z}_+ \quad A^n = A \times A \times \cdots \times A = \{(x_1, \dots, x_n) : x_i \in A\}$$
$$(\mathcal{A} = \{A\} \ f : \{1, \dots, m\} \to \mathcal{A} \ f(i) = A \ i = 1, \dots, m)$$

Definition (sequence set) X set
$x : \mathbb{Z}_+ \to X$ (infinite) sequence, ω-tuple of elements in X
$x = (x(1), x(2), \dots) = (x_1, x_2, \dots) = (x_n)_{n\in\mathbb{N}}$
$X^\omega = \{x : x \text{ sequence in } X\}$

$$\mathcal{A} = \{A_i\}_{i\in\mathbb{Z}_+}$$
$$A_1 \times A_2 \times \cdots = \prod_{i=1}^{\infty} A_i = \prod_{i\in\mathbb{Z}_+} A_i = \{x \in X^\omega \text{ for } X = \bigcup_{i=1}^{\infty} A_i \ x_i \in A_i \ \forall i \in \mathbb{Z}_+\}$$
$$X^\omega = \prod_{i=1}^{\infty} X = \prod_{i\in\mathbb{Z}_+} X$$

(later define $\prod_{\alpha\in J} A_\alpha$ for $\mathcal{A} = \{A_\alpha\}_{\alpha\in J}$)

6. Finite sets and Cardinalities

Definition set A finite $\exists n \in \mathbb{N}$ $\exists f : S_n = \{1, \dots, n-1\} \to A$ bijective
$|A| := n - 1 \in \mathbb{N} \cup \{0\}$ cardinality of A $|A| < \infty$

Caution why is this well defined?

Can $S_m \overset{f'}{\leftrightarrow} A \overset{f}{\leftrightarrow} S_n \quad m \neq n$

to prove is $\nexists f : S_m \to S_n$ bijective if $n \neq m$

This is intuitively clear, but it is good to prove it formally using the tools from set theory we developed. I skip details but it requires some steps which are important for other reasons.

Theorem 6.2 Let A be a set. Assume $\exists f : A \to \{1, \ldots, n\}$ bijective
Let $B \subsetneq A, B \neq \varnothing$.
Then $(1) \exists m \in \mathbb{N}\ m < n\ \tilde{f} : B \to \{1, \ldots, m\}$ bijective
and (2) $\nexists \hat{f} : B \to \{1, \ldots, n\}$ bijective

The proof uses induction

$C = \{n \in \mathbb{Z}_+ : \text{Thereom holds}\}$

prove that C is inductive

$1 \in C$ if $n \in C \Rightarrow n + 1 \in C \Rightarrow C = \mathbb{Z}_+$

Corollary if A is finite, $\nexists$ bijection between A and a proper subset B of itself

pf. if $A \simeq B$, since $A \simeq \{1, \ldots, n\} \Rightarrow B \subsetneq A\ B \simeq \{1, \ldots, n\}$
previous theorem (2) $\Rightarrow \lightning$

Corollary $\mathbb{Z}_+ = \mathbb{N}$ is not finite

pf. $n \mapsto n + 1\ f : \mathbb{Z}_+ \to \mathbb{Z}_+ \backslash \{1\}$ bijective

Corollary $|A|$ is well defined(for A finite)

pf. otherwise, $\{1, \ldots, n\} \simeq \{1, \cdots, m\}\ m < n$

Corollary A finite $B \subsetneq A \Rightarrow B$ fintite and $|B| < |A|$

Corollary The following are equivalent $\forall B \neq \varnothing$
$(1) B$ is finite
$(2) \exists f : S_n \to B$ surjective for some n
$(3) \exists f : B \to S_m$ injective for some m

Corollary Finite unions of finite sets are finite

$\mathcal{A} = \{A_\alpha\}_{\alpha \in J}\ |J| < \infty\ \ \forall \alpha \in J\ |A_\alpha| < \infty$

$\Rightarrow |\bigcup_{\alpha \in J} A_\alpha| = |\bigcup_{A \in \mathcal{A}} A| < \infty$

induction on $|J| = n$

pf. $|J| = 2\ \mathcal{A} = \{A, B\}$

$\{1, \ldots, n\} \leftrightarrow A \qquad \{1, \ldots, m\} \leftrightarrow B$

$\{1, \ldots, n + m\} \xrightarrow{surjective} A \cup B$

induction set $A_n \cup \cdots \cup A_m = (\underbrace{A_n \cup \cdots \cup A_{m-1}}_{A}) \cup \underbrace{A_n}_{B}$

7. Countable and uncountable sets

Definition A set <u>infinite</u> if not finite $\forall n\ \nexists f : S_n \to A$ bijective
A countably infinite if $\exists f : A \to \mathbb{Z}_+$ bijective

Remark countably infinite $\Rightarrow$ infinite
pf. if not $\exists$ bijection(surjection) $\mathbb{Z}_+ \to \{1,\dots,n\} \Rightarrow \exists$ injection $i : \mathbb{Z}_+ \to \{1,\dots,n\}$
$\{1,\dots,n\} \subsetneq \mathbb{Z}_+ \quad I = i(\{1,\dots,n\}) \subsetneq \{1,\dots,n\}$
$i|_{\{1,\dots,n\}} : \{1,\dots,n\} \to I$ bijection
$\{1,\dots,n\}$ has bijection to proper subset $\lightning$

Definition A countable$\Leftrightarrow A$ finite or countably ∞
uncountable otherwise

Lemma $C \subset \mathbb{Z}_+$ infinite $\Rightarrow C$ countalby infinite
pf. construct $h : \mathbb{Z}_+ \xrightarrow{\text{bijective}} C$ with recursive definition

$$h(n) = \overbrace{\text{smallest elemet of }}^{\text{min}} \underbrace{C \backslash h(\{1,\dots,n-1\})}_{C_{n-1}} \to \text{ exists because } \subset \mathbb{Z}_+$$

and prove h is bijective!

Recursive in terms of itself but care is needed e.g. $h(n)$ =minC_{n+1} nonsense since $h(n) \notin C_{n+1}$

<u>Principle of recursive definition</u> (see book §1.8 for more detail; skip here)

If $h(1) \in A$ and $\exists$ formula defining $h(n)$ in terms of $h(1),\dots,h(n-1)$
then this formula determines a unique function $h : \mathbb{Z}_+ \to A$

Theorem Let $B \neq \varnothing$. Then the following are equivalent
(1) B countable (including finite!)
(2) $\exists f : \mathbb{Z}_+ \to B$ surjective
(3) $\exists f : B \to \mathbb{Z}_+$ injective

Corollary B countable, $C \subseteq B \underset{(3)}{\Longrightarrow} C$ countable

Corollary $\mathbb{Z}_+ \times \mathbb{Z}_+$ countably infinite

pf.(1)

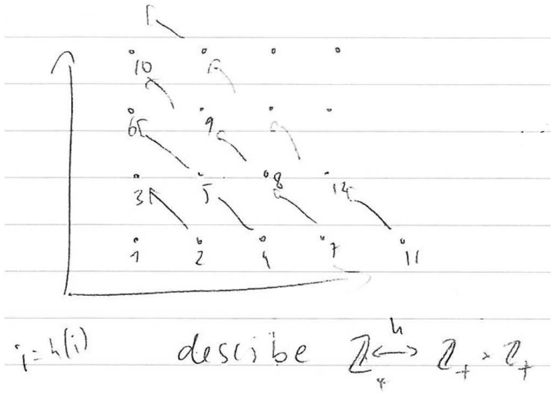

Figure 1 - 5: proof

(2) define $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ by $f(m,n) = 2^m 3^n$ injective
(prime factorization is unique)

Exercise $\mathbb{Q}$ countably infinite

Theorem countable union of countable sets is countable
i.e. $\mathcal{A} = \{A_\alpha\}_{\alpha\in J}$ J countable $\forall\alpha \in J$ A_α countable
$\Rightarrow \bigcup_{A\in\mathcal{A}} A$ countable

pf. fix $g : \mathbb{Z}_+ \to J$ surjective
$\forall\alpha \in J$ $f_\alpha : \mathbb{Z}_+ \to A_\alpha$ surjective
consider $h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{\alpha\in J} A_\alpha$
$h(k,m) = f_{g(h)}(m)$ h surjective

Theorem Finite product of countable set is countable
i.e. $A_1, \dots, A_n$ countable $\Rightarrow \prod_{i=1}^{n} A_i$ countable

Remark not true for ∞ products
$\prod_{i=1}^{\infty}\{0,1\} = \{0,1\}^\infty$ uncountable
$\{0,1\}^\infty \underset{h}{\longleftrightarrow} \mathcal{P}(\mathbb{Z}_+) = \{A : A \subset \mathbb{Z}_+\}$
$h((x_1, x_2, \dots, x_n, \dots)) = \{n \in \mathbb{Z}_+ : x_i = 1\}$
$h^{-1}(A) = (x_1, x_2, \dots)$ with $x_n = \begin{cases} 1 \text{ if } n \in A \\ 0 \text{ if } n \notin A \end{cases}$

Theorem $\forall A$ $\exists$ no bijection between A and $\mathcal{P}(A)$

pf. assume $A \overset{f}{\simeq} \mathcal{P}(A)$
consider $B = \{a \in A : a \notin f(a)\} \subset A$, $B \in \mathcal{P}(A)$
let $b = f^{-1}(B) \in A$
if $b = f^{-1}(B) \in B \Rightarrow b \notin f(b) = f(f^{-1}(B)) = B$
if $b \notin B \Rightarrow b \in f(b) = B$ ↯

pf. can be modified to f surjection $A \longrightarrow \mathcal{P}(A)$ (Exercise)

Theorem(book) $\exists$ no surjective map $A \to \mathcal{P}(A)$
$\exists$ no injective map $\mathcal{P}(A) \to A$

Notation we will write $A \hookrightarrow B$ "A embeds in B" if $\exists\, f : A \to B$ injective

Theorem $\mathbb{R}$ uncountable
pf. describe $\{0,1\}^\omega \underset{h}{\to} \mathbb{R}$ injective

$$(x_1, \dots, x_n, \dots) \mapsto \sum_{i=1}^{\infty} \frac{x_i}{3^i} \qquad \left[\begin{array}{ll} \text{not 2 because} & (0,1,1,1,\dots) \\ & \downarrow \\ & 1/2 \\ & \uparrow \\ & (1,0,\dots,0,\dots) \end{array} \right]$$

now if $\mathbb{R}$ countable
$\exists f : \mathbb{R} \to \mathbb{Z}_+$ injective
$h \circ f : \{0,1\}^\omega \to \mathbb{Z}_+$ injective $\lightning$

But this makes use of algebraic and analytic properties of $\mathbb{R}$ etc.. (convergence...)
later proof only using order properties

Definition If for f, A $\exists f : A \to \mathbb{R}$ bijective say
A has the cardinality $\aleph_0$ of the continuum
linear continuum if bijection preserves order

Example $\{0,1\}^\omega$ has cardinality of the continuum (need a bit of proof!)

Remark

$a)$ $\{A \subseteq \mathbb{Z}_+ \ |A| < \infty\}$ all finite subset of $\mathbb{Z}_+$ is countble!
$\downarrow$
$b)$ $(\mathbb{Z}_+)^\omega_0 = \{(x_1, \dots)\ x_i \in \mathbb{Z}_+ \quad \underbrace{\exists n\ \forall N > n\ x_N = 0}_{\text{eventually zero integer sequence}} \}$ countable!
$\downarrow$ $\mathbb{Z}_+ \to \mathbb{Z}$
$c)$ $\mathbb{Z}[t] = \{$polynomial with integer coefficients$\}$ countable
$\{\text{algebraic numbers}\} = \bigcup_{\mathcal{P} \in \mathbb{Z}[t]} \{\text{roots of } \mathcal{P}\} \subset \mathbb{R}$ countable
$\downarrow$
$d)$ transcendental numbers $= \mathbb{R} \setminus \{\text{algebraic numbers}\}$ are uncountable
$\uparrow$(Cantor's proof of existence of transcendental numbers)
even although very hard to find explicit transcendental numbers
π, e, a^b a rational, b algebraic but not rational
e.g. $2^{\sqrt{2}}$ (Baker's theorem)
but proofs are very hard!!

See Exercise 1.7.6 p.49

Definition Say sets A, B of same cardinality ("equicardinal") $|A| = |B|$,
if $\exists\, f : A \to B$ bijective

Th (Schroeder-Bernstein) $A \hookrightarrow B,\ B \hookrightarrow A \Rightarrow |A| = |B|$.

9. Infinite sets and axiom of choice

Some criteria for infinite sets we had are sufficient to exactly characterize infinite sets

Theorem 9.1 A set, the following are equivalent:
(1) $\exists f : \mathbb{Z}_+ \to A$ injective
(2) $\exists A \leftrightarrow B$ bijective $B \subsetneq A$
(3) $|A| = \infty$

pf. important for us is (3)⇒(1) Assume $|A| = \infty$
construct $f(n)$ by induction
$\exists a_1 \in A$ set $f(1) := a$
$\exists a_n \in \underbrace{A\backslash(\{1,\ldots,n-1\})}_{A_n}$ set $f(n) := a_n$

This proof uses a choice of element in an infinite family of sets $\{A_n\}$
The freedom of such choice does not follow from previous set constructions so we need a new method

Axiom of choice(AC)

$$\mathcal{A} = \{A_i\}_{i\in I} \quad A_i \neq \varnothing \; A_i \cap A_j = \varnothing \quad i \neq j \tag{1}$$
$$\Rightarrow \exists C \subset \bigcup_{i\in I} A_i \text{ with } |C \cap A_i| = 1 \forall i \in I \tag{2}$$

This is the same as saying $\exists f : \mathcal{A} \to \bigcup \mathcal{A}$ with $f(A) \in A \;\; \forall A \in \mathcal{A}$
because given f take C=image(f) which satisfies (2) because of (1)
and given C define $f(A) = x$ for $x \in C \cap A$

One can show (lemma 9.2 in book) that when one uses a choice function we can get rid of the condition $A_i \cap A_j = \varnothing$

Lemma $\mathcal{B}$ family of sets $\exists c : \mathcal{B} \to \bigcup B$
with $c(B) \in B \;\; \forall B \in \mathcal{B}$

now with this one can make proof of (3)⇒(1) in Theorem 9.1 more precese

let $\mathcal{C} = \{A' \subset A : A' \neq \varnothing\}$
take $c : \mathcal{C} \to \bigcup \mathcal{C}$
$f(n) = c((A\backslash f(\{1,\ldots,n-1\}))$...

AC did generate some controversy as to bizarre consequences like the Well-ordering theorem. But now it is widely accepted.

10. Well-ordered sets

Definition $(A, <)$ well ordered if $\exists \varnothing \neq A' \subset A$
smallest element a=min$A' \in A'$ exists

Example $(\mathbb{Z}_+, <)$
$(\mathbb{Z}, <)$ is not well ordered, neither
$([0,1], <) \quad ((0,1), <) \quad (\mathbb{R}, <)$

Construction of well ordered sets

(*a*) if $(A,<)$ well ordered and $B \subset A$
$(B, <|_{B\times B})$ is well ordered

(*b*) A, B well ordered $\Rightarrow A \times B$ with dictionary order is well ordered

Theorem Every nonempty finite ordered set has the order type of $(S_n, <)$
So is well ordered

Example $\mathbb{Z}_+^n$ is well ordered with dictionary order
$\mathbb{Z}_+^\omega$ also has a "dictionary order"
$(a_1, \dots, a_n, \dots) < (b_1, \dots, b_n, \dots)$
if $\exists i \; a_1 = b_1 \; \dots \; a_{i-1} = b_{i-1} \; a_i < b_i$
but it's not well ordered e.g.$\{(1, \dots, 1, \underset{i}{2}, 1, \dots, 1) \; : \; i \in \mathbb{Z}_+\}$
has no smallest element
is there another $<$ making $\mathbb{Z}_+^n$ well ordered?

Theorem (Well-ordering theorem W.O.T.; Zermelo 1904) For every set $A \; \exists <$ such that $(A,<)$ well ordered

This proof only uses the AC and startled many mathematicians at that time which led to suspicions about AC.
Unfortunately the proof (as for the AC) is not constructive, so one can't know what is $<$?!

Corollary $\exists a$ well-ordered uncountable set

Definition X well ordered set $\alpha \in X$ let $S_\alpha = \{x \in X : x < \alpha\}$
section of X by α

(needed later)

lemma $\exists$ well ordered set A with largest element Ω
s.t. $S_\Omega = \{a \in A : a < \Omega\} = A \backslash \{\Omega\}$
is uncountable, but all other sections of A
$S_\alpha = \{a \in A : a < \alpha\}$ are countable

write $A = S_\Omega \cup \{\Omega\} = \overline{S_\Omega}$

Example of something similar
$(\{1\} \cup \{1 - \frac{1}{n} : n \in \mathbb{Z}_+\}, <)$ well ordered
then $\Omega = 1$ and $|S_\Omega| = \infty$ but $\forall a < 1 \;\; |S_a| < \infty$

Theorem if $A \subset S_\Omega$ countable then
A has an upper bound in S_Ω

11. The Maximum Principle

AC has several consequences (later proved equivalent to it) of the type "maximum principle"
Two versions here

Definition A set, $\prec \subset A \times A$ is strict partial order(s.p.o)
if (1) $\forall a \in A \ a \nprec a$ (Nonrefelxivity)
(2) $\forall a, b, c \ a \prec b$ and $b \prec c \Rightarrow a \prec c$ (Transitivity)
(like order but don't need to compare all elements)

Ex. $(\mathcal{P}(A), \subsetneq)$

Remark $a \leq b :\Leftrightarrow a < b$ or $a = b$ defines a partial order(not strict)

Definition $(A, \prec)$ s.p.o. set $B \subset A$ is (simply) → I use the word 'chain'
ordered subset if
$\prec|_{B \times B}$ is an order on B
(i.e. $\forall b, c, \in B: \ b \prec c$ or $c \prec b$)

Ex. $\{S_n : n \in \mathbb{Z}_+\} \subset \mathcal{P}(\mathbb{Z}_+)$ with $\subsetneq$

Definition B is maximal ordered subset (='maximal chain')
if B is ordered subset and $\forall A \supset B' \supsetneq B$, B' is not ordered subset of A

Ex. $\mathcal{A} = \{S_n : n \in \mathbb{Z}_+\}$ is not max ordered
$\cap$
because $\mathcal{A} \cup \{\mathbb{Z}_+\}$ is ordered
$\shortparallel$
but $\mathcal{A}' = \{S_n\} \cup \{\mathbb{Z}_+\}$ is m.o. in $(\mathcal{P}(\mathbb{Z}_+), \subsetneq)$

pf. $A \notin \mathcal{A}'$ $A \subset \mathbb{Z}_+$ assume $\mathcal{A}' \cup \{A\}$ is ordered
if $|A| = \infty$ $A \supset S_n$ $\forall n \Rightarrow A = \mathbb{Z}_+ \in \mathcal{A}'$↯
if $|A| < \infty$ let $m = \max A (\in A)$
then $S_m \subset A$ or $A(\ni m) \subset S_m (\not\ni m)$↯
$\Downarrow$
So $S_m \cup \{m\} \subset A \subset S_{m+1}$
$\shortparallel$ S_{m+1}
$\Rightarrow A = S_{m+1} \in \mathcal{A}'$↯

Theorem(Max Principle, $\Longleftrightarrow$ AC) $(A, \prec)$ s.p.o. set $\Rightarrow \exists B \subset A$ max (simply) ordered subset

Definition $(A, \prec)$ s,p.o. set $B \subset A$ subset
$c \in A$ upper bound for B if $\forall b \in B$ $b = c$ or $B \prec C$
$\Omega \in A$ is maximum element if $\nexists a \in A$ $\Omega \prec a$

Remark if A is ordered, maximum element Ω is unique(if exists), but not always when A is strictly partially ordered

Lemma (Zorn) $(A, \prec)$s.p.o.
$\Updownarrow$ if $\forall B \subset A$ ordered $\exists$ upper bound of B in A
AC $\Rightarrow A$ has a maximum element

This has some important applications:
1 LA: every vector space(infinite dimension) has a basis(next page)
2 functional analysis: Hahn-Banach
$T' : V' \to W$ $V' \subset V$ linear
$\exists T : V \to W$ $T' = T|_{V'}$ with $||T|| = ||T'||$

Definition 3 cardinality

we say $|A| < |B|$ B has greater cardinality than A

if $\exists c : A \to B$ injective $\nexists \mathcal{K} : B \to A$ injective

$$\text{AC} \Updownarrow \qquad\qquad \text{AC} \Updownarrow$$

$\exists c' : B \to A$ surjective $\nexists \mathcal{K}' : A \to B$ surjective

we say $|A| = |B|$ B has same cardinality as A

if $\exists c : A \to B$ bijective

Theorem For any two sets A, B

$\Updownarrow$ either $|A| < |B|$, $|A| = |B|$ or $|B| < |A|$

W.O.T. (Ex. 1.10.11) Cardinalities are "strictly ordered"

recall $|\mathbb{Z}_+| = \omega$ infinitely countable cardinality

$|\mathbb{R}| = \aleph_0$ "aleph zero" cardinality of the continuum

Continuum hypothesis $\nexists$ cardinality κ with $\kappa > \omega$, $\kappa < \aleph_0$

Generalized C. h. $\forall |A| = \infty$, $\nexists$ cardinality $\kappa > |A|$, $\kappa < |\mathcal{P}(A)|$

independent of other set theory axioms!

Theorem V VS over F V has a basis

pf. (use Maximum Priniciple)

$\mathcal{K} = \{A \subset V : A \text{ linearly independent}\}$

$(\mathcal{K}, \subsetneq) \subset \mathcal{P}(V)$ is s.p.o. set

$\exists$ max chain $\mathcal{E} \subset \mathcal{K}$ (chain = (simply) ordered subset)

Consider $B = \bigcup_{E \in \mathcal{E}} E \subset V$

1) we claim B linearly independent

$\lambda_i \in F$ $\sum_{i=1}^{n} \lambda_i v_i = 0$ $v_i \in E_i$ $E_i \in \mathcal{E}$

$\mathcal{E}$ chain $\Rightarrow \exists j$ $E_j = \max(E_i)_{i=1}^n \supset E_i$ $\forall i$

$v_i \in E_j$ $E_j \in \mathcal{E} \subset \mathcal{K}$ linearly independent $\Rightarrow \lambda_j = 0$

2) B generating

Assume B not generating. $\exists v \notin \operatorname{span}(B)$ $v \in V$

$\Rightarrow B \cup \{v\} (\in \mathcal{K})$ linearly independent $B \cup \{v\} \supset E$ $\forall E \in \mathcal{E}$

$\mathcal{K} \supset \mathcal{E}' = \mathcal{E} \cup \{\mathcal{B} \cup \{\text{v}\}\} \supset \mathcal{E}$ is a chain in $(\mathcal{K}, \subsetneq)$

$\mathcal{E}$ is maximal $\Rightarrow \mathcal{E}' = \mathcal{E} \Rightarrow B \cup \{v\} \in \mathcal{E} \Rightarrow$

$v \in \bigcup_{E \in \mathcal{E}} E = B \Rightarrow \lightning$

$\Rightarrow B$ basis

Remark highly non-constructive:

What is a basis of $\mathbb{R}$ over $\mathbb{Q}$?