Chapter 2. Topological Spaces and Continuous Functions

12. Topological Spaces, topological operations, separation

Definition X set, $\mathcal{A} \subset \mathcal{P}(X)$ topoogy on X if

(1) $\varnothing, X \subset \mathcal{A}$

(2) closed under arbitrary union

$\mathcal{A}' \subset \mathcal{A} \, : \, \bigcup \mathcal{A}' = \bigcup\limits_{A' \in \mathcal{A}'} A' \in \mathcal{A}$

(3) closed under simple intersection

$A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

Remark (3) $\Leftrightarrow$ $\forall \mathcal{A}' \subset \mathcal{A} \; |\mathcal{A}'| < \infty \; \bigcap \mathcal{A}' \in \mathcal{A}$

closed under finite $\bigcap$

Definition $(X, \mathcal{A})$ topology call $A \in \mathcal{A}$ an open set

Ex. $\mathcal{A} = \mathcal{P}(X)$ discrete topology write $(X, \mathcal{P}(X)) = X_{\text{discr}}$

$\mathcal{A} = \{\varnothing, X\}$ trivial(indiscrete) topology

Ex. $\mathcal{A} =$ maximal chain in $(\mathcal{P}(X), \subsetneq) \to$ HW

Ex. $\mathcal{A} = \{A \subset X : |X \backslash A| < \infty\} \cup \{\varnothing\}$

finite complement (f.c.) topology

similarly: countable complement topology

trivial topology motivates

Definition $(X, \mathcal{A})$ topological space

$x_1, x_2 \in X$ are call topologically indistinguishable

if $\forall A \in \mathcal{A} \quad (x_1 \in A \Leftrightarrow x_2 \in A)$

$\downarrow$ means

$x_1, x_2 \in A$ or $x_1, x_2 \notin A$

topology $\mathcal{A}$ cannot distinguish x_1, x_2

Definition (T_0 Kholmogorov axiom) - (very basic separation axiom)

$\mathcal{A}$ is T_0 if $\nexists$ topologically indistinguishable points

$\Leftrightarrow \forall x_1 \neq x_2 \; \exists A \in \mathcal{A} \; x_1 \in A \; x_2 \notin A$ or $x_1 \notin A \; x_2 \in A$

Figure 2 - 1

Let $\mathcal{A}$ be a topology on X. Define an equivalence relation on X by

$x_1 \sim x_2 \Leftrightarrow x_1$ is topologically indistinguishable from x_2

Then let $\qquad\qquad\qquad\qquad \downarrow$ equivalence class of x under $\sim$

$$\tilde{X} = X/\sim = \{[x]_\sim : x \in X\}$$

and define a topology $\tilde{\mathcal{A}}$ on $\tilde{X}$ by

$$\tilde{\mathcal{A}} = \{\{[x]_\sim : x \in A\} : A \in \mathcal{A}\}$$

thus topology identifies (and removes) topologically indistinguishable points
$(\tilde{X}, \tilde{\mathcal{A}})$ is called Kholmogorov quotient of $(X.\mathcal{A})$
Trivial topology is not T_0 unless $|X| = 1$

Ex. if $\mathcal{A} = \{\varnothing, X\}$ then $\tilde{X} = \{*\}$

often will assume T_0

Definition $(X, \mathcal{A})$ $A \subset X$ is called closed if $X \backslash A$ is open

$\{A_i\}_{i \in I}, A_i$ closed, $\bigcap\{A_i\} = \bigcap\limits_{i\in I} A_i$ closed $\quad A, B$ closed $\quad A \cup B$ closed

Definition $x \in X$. An $O \in \mathcal{A}$, $O \ni x$ is called neighborhood of x

Definition $(X, \mathcal{A})$ $A \subset X$ any set
Let $\text{Int}(A) = \bigcup\{A' \in \mathcal{A} : A' \subset A\}$
the interior of A $\{x \in X (x \in A) : \exists O \in \mathcal{A}\ x \in O \subset A\}$

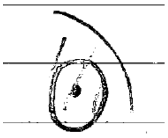


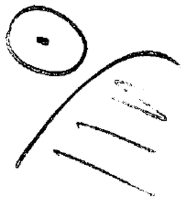
Figure 2 - 2. interior point

$$= \{x \in X : \exists O \in \mathcal{A} \quad x \in O \subset A\}$$

lemma
1) $\text{Int}(A) \subset A$
2) $\text{Int}(A) = A \Leftrightarrow A$ is open ($\Rightarrow \text{Int}(\varnothing) = \varnothing$ $\text{Int}(X)$= X)
3) $A \subset B \Rightarrow \text{Int}(A) \subset \text{Int}(B)$
4) $\text{Int}(\text{Int}(A))$ =$\text{Int}(A)$($\Leftrightarrow$ $\text{Int}(A)$ open)

pf.
1) ✓
2) "$\Leftarrow$" open $\quad A \in \mathcal{A}' = \{A' \in \mathcal{A} : A' \subset A\}$
$A \supset \text{Int}(A) = \bigcup \mathcal{A}' \supset A \ \Rightarrow \text{Int}(A) = A$
$\uparrow 1)$
"$\Rightarrow$ " $\text{Int}(A) = A \quad A = \bigcup\{\underbrace{A' \in \mathcal{A} : A' \subset A}_{\mathcal{A}'}\}$
$\mathcal{A}' \subset \mathcal{A} \xRightarrow{\text{top. prop.}} A \in \mathcal{A} \Rightarrow A$ open
3) $A \subset B$ $\mathcal{A}' = \{A' \in \mathcal{A} : A' \subset A\}$
$\subset \mathcal{B}' = \{B' \in \mathcal{A} : B' \subset B\}$
$\text{Int}(B) = \bigcup \mathcal{B}' \supset \bigcup \mathcal{A}' = \text{Int}(A)$
4) $\text{Int}(A) = \bigcup \mathcal{A}'$ for $\mathcal{A}' \subset \mathcal{A} \Rightarrow \text{Int}\ (A)$ open
$\overset{2)}{\Longrightarrow} \text{Int}(\text{Int}(A))$=$\text{Int}(A)$

Definition $(X,\mathcal{A})$ $A\subset X$ define the exterior← try not to use much
$\mathrm{Ext}(A) = \mathrm{Int}(X\backslash A)$
$= \{x\in X : \exists O\in\mathcal{A}\quad O\cap A = \varnothing\}$



x ∈Ext(A) exterior point

Figure 2 - 3. exterior point

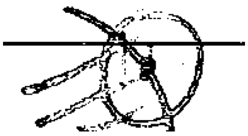
closure

$$\overline{A} = X\backslash\mathrm{Ext}(A) = X\backslash\mathrm{Int}(X\backslash A)$$
$$= \{x\in X : \forall O\in\mathcal{A}\ O\ni x \Rightarrow O\cap A\neq\varnothing\}$$

Lemma Properties of closure
$\overline{A}\supset A\quad \overline{A} = A \Leftrightarrow A$ closed$(\Rightarrow \overline{\varnothing} = \varnothing, \overline{X} = X)$
$\overline{\overline{A}} = \overline{A}$ (i.e. $\overline{A}$ is closed)
$A\subset B\Rightarrow \overline{A}\subset\overline{B}$

Remark V VS $S\subset V$ set span has similar properties:
span$(S)\supset S$
$S\supset S' \Rightarrow$ span$(S)\supset$ span(S')
span(span(S))=span(S)
Operations of this sort are called hull operations
[Similar conv(X) convex hull]
so closure is a hull operation

Definition $A\subset X$ Bd $A = \overline{A}\cap\overline{X\backslash A} = \overline{A}\backslash\mathrm{Int}(A)$
elsewhere is called (top.) boundary of A
$(\partial A!)$ Bd $A = \{x\in X : \forall O\in\mathcal{A}\ O\in x\ O\not\subset A\ O\cap A\neq\varnothing\}$
$x\in$ Bd A is called boundary point



x ∈Bd(A) boundary point

Figure 2 - 4. boundary point

Ex. $\mathcal{A} =$ discrete top. all $A\subset X$ open $\Rightarrow$ and closed
$\overline{A} = \mathrm{Int}(A) = A$ Bd $A = \overline{A}\backslash\mathrm{Int}(A) = \varnothing$

Ex. $\mathcal{A} =$ trivial top. $A\subset X$ $A\neq\varnothing, X$
$\mathrm{Int}(A) = \varnothing$ $\overline{A} = X$ Bd $A = X$

Definition we say $(X, \mathcal{A})$ is T_1 (Fréchet)
if $\forall x_1 \neq x_2\ \exists\ O_{1,2} \in \mathcal{A}\ :\ O_1 \ni x_1\ O_1 \not\ni x_2$ and $O_2 \ni x_2\ O_2 \not\ni x_1$

Figure 2 - 5. T_1 axiom

Lemma $T_1 \overset{(1)}{\Longleftrightarrow}$ points are closed $\overset{(2)}{\Longleftrightarrow}$ finite sets are closed
$\overline{\{x\}} = \{x\}$

pf. (2) because closedness is invariant under finite union
(1) $\mathcal{A}$ is T_1 let $x_1 \in X\ x_2 \neq x_1$
$\exists x_2 \ni O_2$ open $O_2 \notin x_1 \quad O_2 \cap \{x_1\} = \varnothing$
$\Rightarrow x_2 \notin \overline{\{x_1\}} \quad \forall x_2 \neq x_1$
$\Rightarrow \overline{\{x_1\}} = \{x_1\}$

$T_1 \Rightarrow T_0$ but not converse

Ex. $X = \{1, 2\}$

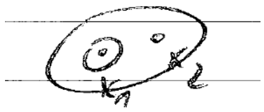


Figure 2 - 6

$\mathcal{A}$ is $T_0 \quad \overline{\{x_1\}} = \{x_1, x_2\}$

Ex. finite complement(f.c.) topology is T_1

Definition $(X, \mathcal{A})$ is T_2 (Hausdorff)
if $\forall x_1, x_2 \in X\ x_1 \neq x_2\ \exists O_1 \ni x_1\ O_2 \ni X_2\ O_i$ open
$O_1 \cap O_2 = \varnothing$

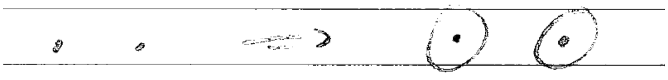


Figure 2 - 7

$T_2 \Rightarrow T_1 (\Rightarrow T_0)$ but not converse

Ex. $|X| = \infty,\ \mathcal{A} = $ f.c. top on X
$x_1, x_2 \in X \quad x_1 \neq x_2$
$O_1 \ni x_1\ O_2 \ni x_2 \quad O_i \in \mathcal{A}$
$O_1, O_2 \neq \varnothing \Rightarrow |X \backslash O_1|, |X \backslash O_2| < \infty$
$|X \backslash (O_1 \cap O_2)| = |(X \backslash O_1) \cup (X \backslash O_2)| < \infty$
$|X| = \infty \Rightarrow O_1 \cap O_2 \neq \varnothing$ f.c. top. is not Hausdorff

Definition $(X,\mathcal{A}'),(X,\mathcal{A})$ two topologies on same space
we say $\mathcal{A}'$ is finer than $\mathcal{A}$ if $\mathcal{A}' \supset \mathcal{A}$
strictly finer than $\mathcal{A}' \supsetneq \mathcal{A}$
coarser than $\mathcal{A}' \subset \mathcal{A}$
strictly coarser than $\mathcal{A}' \subsetneq \mathcal{A}$
! to usage of "weaker/stronger","higher/smaller" elsewhere (try to avoid)

Remark of course if $\mathcal{A}$ is T_i $(i = 0,1,2)$
$\mathcal{A}'$ finer $\Rightarrow \mathcal{A}'$ is T_i, too

Remark $\mathcal{A}' \supset \mathcal{A}$ $A \subset X$ $\overline{A}^{\mathcal{A}'} \subset \overline{A}^{\mathcal{A}}$
$\operatorname{Int}_{\mathcal{A}'}(A) \supset \operatorname{Int}_{\mathcal{A}}(A)$

Definition $A \subset X$ is dense if $\overline{A} = X$.
X is separable if $\exists A \subset X$ dense and countable (p.189-190 in book)

13. Basis for Topology

most important topologies. we will work with can be defined through a basis

Definition X set $\mathcal{B} \subset \mathcal{P}(X)$ is a basis for topology on X if
B1) $\forall x \in X \quad \exists B \in \mathcal{B} \quad x \in B \quad \Leftrightarrow \bigcup \mathcal{B} = X$
B2) $\forall B_1, B_2 \in \mathcal{B} \quad \forall x \in X$
$x \in B_1 \cap B_2 \quad \exists B_3 \in \mathcal{B} \quad x \in B_3 \subset B_1 \cap B_2$

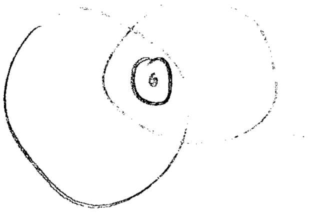


Figure 2 - 8

Definition $X = (X,\mathcal{A})$. $\mathcal{B}$ is basis for topology $\mathcal{A}$
the topology $\mathcal{A}$ is generated by $\mathcal{B}$ $\quad \mathcal{A} = \mathcal{A}(\mathcal{B})$
$A \in \mathcal{A} \Leftrightarrow \forall x \in A \ \exists B \in \mathcal{B} \ x \in B \ B \subset A \qquad (*)$

Remark $\mathcal{B} \subset \mathcal{A}$

Lemma $A \in \mathcal{A} \Leftrightarrow \quad A = \bigcup\{B \in \mathcal{B} : B \subset A\}$

pf. $\Rightarrow \quad A \supset \bigcup\{B \in \mathcal{B} : B \subset A\} \quad A \subset \bigcup\{B \in \mathcal{B} : B \subset A\}$ by $(*)$
$\Leftarrow \quad A = \bigcup\{B \in \mathcal{B} : B \subset A\}$
$\Downarrow$
$\forall x \in A \ \exists B \in \mathcal{B} \ B \subset A \ B \ni x$
$\Downarrow$
$A \in \mathcal{A}$

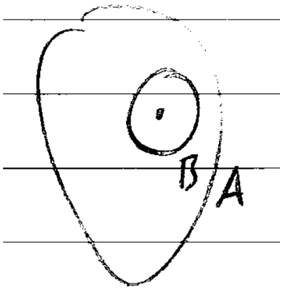


Figure 2 - 9

Lemma $A \in \mathcal{A} \Leftrightarrow \quad \exists \mathcal{B}' \subset \mathcal{B} \quad A = \bigcup \mathcal{B}'$

$\Rightarrow \quad \mathcal{B}' = \{B \in \mathcal{B} : B \subset A\}$

$\Leftarrow \quad A = \bigcup \mathcal{B}' \quad \forall x \in A \ \exists B \in \mathcal{B}' \ x \in B \ B \subset \bigcup \mathcal{B}' = A$

$\Rightarrow A \in \mathcal{A}$

so open sets are unions of basis elements

Lemma $\mathcal{A}(\mathcal{B})$ is a topology

pf. $\mathcal{A}(\mathcal{B}) = \{\bigcup \mathcal{B}' : \mathcal{B}' \subset \mathcal{B}\}$

$\mathcal{B}' = \varnothing \Rightarrow \varnothing \in \mathcal{A}$

$\mathcal{B}' = \mathcal{B}$ use B1) $\Rightarrow X \in \mathcal{A}$

$\{A_\alpha\}_{\alpha \in I} \subset \mathcal{A} \quad A_\alpha = \bigcup \mathcal{B}_\alpha \ \mathcal{B}_\alpha \subset \mathcal{B}$

$$\bigcup_\alpha A_\alpha = \bigcup_\alpha \bigcup \mathcal{B}_\alpha = \bigcup \underbrace{\bigcup_\alpha \mathcal{B}_\alpha}_{\subset \mathcal{B}} \in \mathcal{A}$$

assume $A', A'' \in \mathcal{A}$

$A' = \bigcup \mathcal{B}' \quad A'' = \bigcup \mathcal{B}'' \quad \mathcal{B}', \mathcal{B}'' \in \mathcal{B}$

$$\text{let } x \in A' \cap A'' \Rightarrow \begin{array}{l} \exists B' \in \mathcal{B}' \quad x \in B' \cap B'' \\ \ \ B'' \in \mathcal{B}'' \\ \qquad\quad \cap \qquad\quad B' \subset A' \\ \qquad\quad \mathcal{B} \qquad\quad B'' \subset A'' \end{array}$$

$\xRightarrow{\text{B2)}} \exists B''' \in \mathcal{B} \subset \mathcal{A} \quad x \in B''' \subset B' \cap B''$

$x \in B''' \subset A' \cap A''$

$\forall x \in A' \cap A'' \quad \exists B''' \in \mathcal{A} \quad x \in B''' \subset A' \cap A''$

$\Rightarrow A' \cap A''$ is open $\Rightarrow A' \cap A'' \in \mathcal{A}$

Lemma (B.2) $(X, \mathcal{A}) \quad \mathcal{C} \subset \mathcal{A}$

how to identify a basis $\quad \forall O \in \mathcal{A} \quad x \in O \quad \exists U \in \mathcal{C} \quad x \in U \subset O$

$\Rightarrow \mathcal{C}$ is a basis

Lemma $(\overset{\mathcal{B}}{X, \mathcal{A}}), (\overset{\mathcal{B}'}{X, \mathcal{A}'})$

$\mathcal{A}' \supset \mathcal{A} \Leftrightarrow \forall x \in X \quad \forall B \in \mathcal{B}$

$\mathcal{A}'$ finer $\quad x \in B \Rightarrow \exists B' \in \mathcal{B}' \quad x \in B' \subset \mathcal{B}$

Definition $(\mathbb{R},<)$ $\mathcal{B}=\{(a,b):a<b\}$ $(a,b)=\{x:a<x<b\}$
topology generated by $\mathcal{B}$ is called standard (Euclidean) topology on $\mathbb{R}$
assumed below by default

Definition $\mathcal{B}'=\{[a,b):a<b\}$
$\mathbb{R}_\ell=(\mathbb{R},\mathcal{A}(\mathcal{B}'))$ lower limit topology

Lemma $(X,\mathcal{A})$ basis $\mathcal{B}$ $\operatorname{Int}(A)=\bigcup\{B\in\mathcal{B}:B\subset A\}$

Ex. 1) $A=(0,1)$ $A\subset\mathbb{R}$
$\overline{A}=[0,1]$ $\operatorname{Int}(A)=A=(0,1)$ (A open)
2) $A=\{1\}$ $\overline{A}=A$ $\operatorname{Int}(A)=\varnothing$
$\operatorname{Bd} A=A$
3) $A=[0,1)$ $\overline{A}=[0,1]$ $\operatorname{Int}(A)=(0,1)$
$\operatorname{Bd} A=\{0,1\}$
4) $A=\mathbb{Q}$ $\exists a',a''\in(a,b)$ $a'\in\mathbb{Q}$ $a''\in\mathbb{R}\setminus\mathbb{Q}$
$\overline{A}=\mathbb{R}$ $\operatorname{Int}(A)=\varnothing$
$\operatorname{Bd}(A)=\mathbb{R}$ rational and irrational numbers are <u>dense</u>

Ex. $\mathbb{R}_\ell$
1) $A=[0,1]$ $\operatorname{Int}(A)=[0,1)$
$\overline{A}=A$ $\operatorname{Bd} A=\{1\}$
2) $A=[0,1)$ $\operatorname{Int} A=A=[0,1)$ A open and closed!
$\overline{A}=A$ $\operatorname{Bd} A=\varnothing$
3) $A=\{1\}$ $\operatorname{Int} A=\varnothing$
$\overline{A}=A$ $\operatorname{Bd} A=\{1\}$
4) $A=\mathbb{Q}$ like previous example

Definition X set $\mathcal{S}$ subbasis $\mathcal{S}\subset\mathcal{P}(X)$
topology of X is the topology with basis $\bigcup\mathcal{S}=X$
$\mathcal{B}=\{\bigcap\mathcal{S}':\mathcal{S}'\subset\mathcal{S}\ |\mathcal{S}'|<\infty\}$
finite intersections of elements in $\mathcal{S}$
$\mathcal{A}$ = unions of <u>finite</u> intersections of elements in $\mathcal{S}$

14. The Order Topology

First major source of important topologies!

Definition $(X,<)$ ordered set $a<b$
$(a,b)=\{x:a<x<b\}$
$[a,b)=\{x:a\le x<b\}$ etc. (had before)

Definition $(X,<)$ ordered set. the order topology $\mathcal{A}$ on X
is the one with basis $\mathcal{B}$ consisting of
1) all open intervals (a,b) $a<b$
2) $[a_0,b)$ a_0 smallest element $b>a_0$ (if $\exists a_0$!)
$(a,b_0]$ b_0 largest element $a<b_0$ (if $\exists b_0$!)

Ex. $|X| < \infty$ X ordered $\Rightarrow \mathcal{A}$ discrete topology
similarly $(\mathbb{Z}_+, <)$

Ex. $(\mathbb{R}, <)$ $\mathcal{A} =$ Euclidean topology on $\mathbb{R}$

Ex. $\mathbb{R} \times \mathbb{R}$ with dictionary order

Recall $a \times b > c \times d \Leftrightarrow a > c$ or $(a = c$ and $b > d)$
no smallest or largest element

$$(c \times d, a \times b) = \begin{cases} \{c\} \times (d, b) & a = c \\ \{c\} \times (d, \infty) \cup (c, a) \times \mathbb{R} \cup \{a\} \times (-\infty, b) & c < a \end{cases}$$

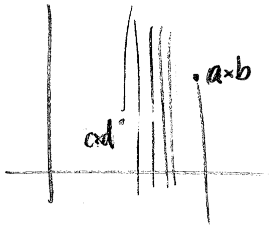


Figure 2 - 10

Ex. $\{1, 2\} \times \mathbb{Z}_+$ dictionary order
$1 \times n = a_n$ $2 \times n = b_n$ $X = a_1 \dots a_n \dots b_1\ b_2 \dots$
$(b_1, b_3) = \{b_2\}$ $\{b_2\}$ open
$\{a_1\} = [a_1, a_2)$ $\{a_1\}$ open
$\uparrow$ nearest element
similarly $\{a_i\}$ open $i > 1$
$\{b_j\}$ open $j > 2$
but $\{b_1\}$ is not open

if $b_1 \in B$ open $\exists b_1 \in (a, b) \subset B$ $b > b_1$
$a < b_1$ $a = a_n$ for some n
$B \supset (a, b) \supset \{a_{n+1}, a_{n+2}, \dots\}$
$|B| = \infty$

Definition $(X, <)$ $a \in X$ define $(a, +\infty) = (a, \infty) = \{x : x > a\}$
$[a, \infty) = \{x : x \geq a\}$
$(-\infty, a)$ $(-\infty, a]$ rays
(a, ∞) $(-\infty, a)$ open rays (open sets)
$[a, \infty)$ $(-\infty, a]$ closed rays (closed sets)

15. The Product Topology

Definition $(X, \mathcal{A}), (X, \mathcal{B})$ define a topology
$\mathcal{A} \times \mathcal{B} = \mathcal{C}$ on $X \times Y$ by the basis
$\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$
$\mathcal{C}$ is called the product topology on $X \times Y$

Theorem if $\mathcal{A}' \subset \mathcal{A}$ is a basis for $\mathcal{A}$
$\mathcal{B}' \subset \mathcal{B}$ is a basis for $\mathcal{B}$
$\Rightarrow \{A' \times B' : A' \in \mathcal{A}', B' \in \mathcal{B}'\}$
is a basis for $\mathcal{A} \times \mathcal{B}$

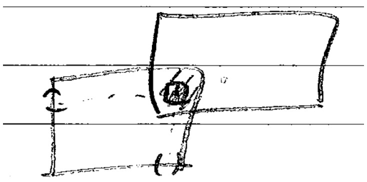


Figure 2 - 11

Definition $\pi_1 : X \times Y \to X \qquad \pi_1(x \times y) = x$
$\pi_2 : X \times Y \to Y \qquad \pi_2(x \times y) = y$
projection

Theorem $\mathcal{S} = \{\pi_1^{-1}(U) : U \subseteq X \text{ open}\} \cup \{\pi_2^{-1}(U) : U \subseteq Y \text{ open}\}$
$(\pi_1^{-1}(U) = U \times Y,\ \pi_2^{-1}(U) = X \times U)$
is a subbasis for $\mathcal{A} \times \mathcal{B}$

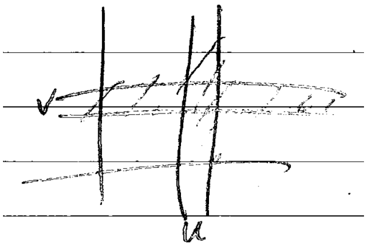


Figure 2 - 12

16. The Subspace Topology

$(X, \mathcal{A})$ topological space $Y \subset X$ ↓ my notation
$(Y, \mathcal{A}')$ with $\mathcal{A}' = \{A \cap Y : A \in \mathcal{A}\} = \mathcal{A}_Y$ is called subspace topology

Lemma $(X, \mathcal{A})$ basis $\mathcal{B}$ $\quad Y \subset X$
$\{B \cap Y : B \in \mathcal{B}\}$ is a basis of $(Y, \mathcal{A}_Y)$

Definition if $X \supset Y \supset U$
we can say $\quad U$ open in Y if $U \in \mathcal{A}_Y$
U closed in Y if $Y \setminus U \in \mathcal{A}_Y$

Ex. $X = \mathbb{R} \quad Y = [0,1] \quad U = (0,1]$
$U = Y$ is open in Y
Y is not open (in X)
U is not open in X

Lemma $X \supset Y \supset U$ is open in Y, Y open in X
$\Rightarrow U$ open in X

Theorem $A \subset X$ $B \subset Y$ $(X \times Y, \mathcal{C})$ $\mathcal{C} = \mathcal{A} \times \mathcal{B}$
$\mathcal{A}$ $\mathcal{B}$ product topology $\mathcal{A}_A \times \mathcal{B}_B = \mathcal{C}_{A \times B}$

Ex. $I = [0,1]$
$I_0^2 = I \times I$ with dictionary order topology will be called the ordered square
$I_0^2 = I \times I \subset \mathbb{R}^2$ but topology on I_0^2 is different from subspace topology of $\mathbb{R}^2$

Ex. $A = \{1/2\} \times (0,1) = (1/2 \times 0, 1/2 \times 1)$ is open in I_0^2
Now consider topology related to $\mathbb{R}^2$
Let $p = 1/2 \times 1/2 \in A$
assume $\exists B \ni 1/2 \times 1/2$ $B \in \mathcal{B}$
$\{(a,b) \times (c,d) : \cap [0,1]^2\}$ basis of relative top.
$\Rightarrow a < 1/2$ $b > 1/2 \Rightarrow B \notin A \Rightarrow p \notin \mathrm{Int}(A)$
In fact, similarly you see $\mathrm{Int}(A) = \varnothing$

Definition $(X, <)$ $Y \subset X$ convex
$\Leftrightarrow \forall a, b \in Y$ $a < b$
$\forall x \in X$ $a < x < b \Rightarrow x \in Y$

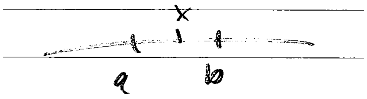


Figure 2 - 13

(Distinguish from $A \subset V$ convex V VS over $\mathbb{R}$)

Remark Intervals and rays are convex
but not the other way around.

Ex. $X = \mathbb{R} \backslash \{0\}$
$Y = (-\infty, 0)$
this is convex subset in X
but one can't make $Y = (-\infty, a)$ $(-\infty, b]$ $(a,b) \ldots$
for a, b in X (not in $\mathbb{R}$!)
$(<' := < |_{Y \times Y})$

$$\begin{tikzcd}
(X,<) \arrow[r, "Y \subset X", "\text{restricted order}"'] \arrow[d] & (Y, < |_{Y \times Y}) \arrow[d] \\
(X, \mathcal{O}_<) \arrow[r, "\text{relative topology}"'] & (Y, \mathcal{O}_{<'})
\end{tikzcd}$$

order topology

Theorem if $Y \subset X$ convex, then $\mathcal{O}_{<|_{Y \times Y}} = (\mathcal{O}_<)_Y$

Convention $(X, <)$ $Y \subset X$ assumed with subspace topology$(\mathcal{O}_<)_Y$
$(= \mathcal{O}_{<|_{Y \times Y}}$ if Y convex!) (see pg. 25)

Ex. $\mathbb{Q} \subset \mathbb{R}$ subspace topology from $(\mathbb{R}, <)$
has a basis $\mathcal{B}_1 = \{(a,b) \cap \mathbb{Q} : a, b \in \mathbb{R}\}$
order topology from $(\mathbb{R}, <)$ has a basis
$\mathcal{B}_2 = \{(a,b) : a, b \in \mathbb{Q}\}$

These bases are not the same (e.g. $\mathbb{Q} \cap (0, \pi) \in \mathcal{B}_1 \setminus \mathcal{B}_2$)
of course $\mathcal{B}_2 \subset \mathcal{B}_1$
and you can show all sets in $\mathcal{B}_1$ are unions of sets in $\mathcal{B}_2$, so the topology of $\mathcal{B}_1, \mathcal{B}_2$ are the same
but this shows for general ordered sets be careful (we will see later examples that top's different)

17. Closed sets, Accumulation points and Limit points

already defined closed sets

Theorem $\varnothing, X$ closed
A, B closed $\Rightarrow A \cup B$ closed
$\{A_i\}_{i \in I}$ closed $\Rightarrow \bigcap_i A_i$ closed

Theorem $X \supset Y \supset A$ A closed in Y $\Leftrightarrow \exists A' \subset Y$ closed $A = A' \cap Y$

Theorem $X \supset Y \supset Z$ Y closed in X, Z closed in Y
$\Rightarrow Z$ closed in X

$A \subset X$ $\overline{A} = \overline{A}^{\mathcal{A}} = \{x \in X \quad \forall O \in \mathcal{A} : O \ni x \Rightarrow O \cap A \neq \varnothing\}$
! $= \{x \in X \quad \forall O \in \mathcal{B} : O \ni x \Rightarrow O \cap A \neq \varnothing\}$
$\mathcal{B}$ basis of $\mathcal{A}$

Theorem $Y \subset X$ $A \subset Y$ $(Y, \mathcal{A}_Y)$ relative topology
$\overline{A}^{\mathcal{A}_Y} = \overline{A}^{\mathcal{A}} \cap Y$

Definition $A \subset X$ $x \in X$ if $x \in \overline{A \backslash \{x\}}$
$(\Leftrightarrow \forall O \subset \mathcal{A}$ $O \ni x \Rightarrow O \cap (A \backslash \{x\}) \neq \varnothing)$
call x an accumulation point(적립점) of A
if $x \in A$ and x is not an accumulation point of A
call x an isolated point(고립점) of A
A is discrete(이산집합) if all its points are isolated

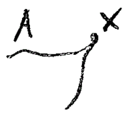


Figure 2 - 14: accumulation point

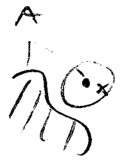


Figure 2 - 15: isolated point

Definition $A_{\text{acc}} = \{\text{acc. points of } A\}$

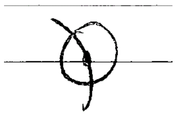


Figure 2 - 16: accumulation point set

Theorem $\overline{A} = A \cup A_{\text{acc}}$

Theorem X is T_1. Then $x \in X$ is accumulation point of A
$\Leftrightarrow$ for every neighborhood $O \ni x$ $|O \cap A| = \infty$

Convergence care is needed with limit and convergence

Definition $X \ni x_1, x_2, \dots, x_n, \dots$
we say $\lim\limits_{n\to\infty} x_n = x \quad x_n \to x$
if $\forall O \ni x \; O \in \mathcal{A} \; \exists N_0 \; \forall n > N_0 \; x_n \in O$

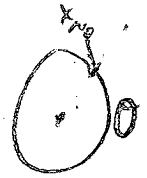


Figure 2 - 17

Remark $(X,\mathcal{A}),\ (X,\mathcal{A}'),\ \mathcal{A}' \supset \mathcal{A} \quad x_n \to_{\mathcal{A}'} x \Rightarrow x_n \to_{\mathcal{A}} x$

Ex. $x_n = n$ in $\mathbb{R}$ with finite complement topology $\mathcal{O}$
Let $a \in \mathbb{R} \quad O \ni a \quad O \neq \varnothing$
$\Rightarrow |R \backslash O| < \infty \quad \exists N \; \forall n \geq N \; x_n \in O$
$x_n \to a$ so $x_n = n \to a \; \forall a \in \mathbb{R}$

Theorem X is T_2 (Hausdorff)
$\Rightarrow$ limit(if it exists!) is unique ↓ unique limit property
i.e. every sequence of points converges to at most one point

Remark Converse is false, see following Ex.

Ex. of a non-Hausdorff space with unique limit property $X = \underbrace{S_\Omega \cup \{\Omega\}}_{\overline{S}_\Omega} \cup \{\Omega'\}$ "duplicate" Ω

$$\mathcal{A} \text{ with top basis } \left\{ \begin{array}{ll} [a_0, b) & \\ (a, \Omega] & a, b \in \overline{S}_\Omega \\ (a, b) & a < b \end{array} \right\} \cup \{(a, \Omega) \cup \{\Omega'\} : a \in S_\Omega\}$$

basis of order topology on $\overline{S}_\Omega$

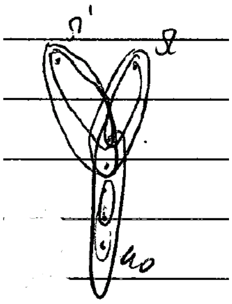


Figure 2 - 18

$\mathcal{A}_{\overline{S}_\Omega}$ = order topology on $\overline{S}_\Omega$
$\mathcal{A}_{X \setminus \{\Omega\}} \simeq$ order topology on $\overline{S}_\Omega$ with $\Omega \to \Omega'$ ($\Rightarrow T_2$)

X is not T_2 because any two basis elements containing Ω, Ω' intersect
but X has the unique limit property
Let $\{x_n\} \subset X$

Lemma $x_n \to \Omega(\text{respectively } \Omega') \Leftrightarrow \exists N \ \forall n \geq N \ x_n = \Omega(\text{respectively } \Omega')$
pf. Assume $\exists$ infinitely many $x_n \in S_\Omega \cup \{\Omega'\}$
if infinitely many $x_n = \Omega' \Rightarrow \exists x_{n_k} \equiv \Omega'$ but $\exists U$ open $\Omega \in U \ \Omega' \notin U$
$\Rightarrow x_n \not\to \Omega \lightning$
So $\exists$ infinitely many $x_n \in S_\Omega$ $\{x_{n_k}\} \subset S_\Omega$ countable
property of $S_\Omega \Rightarrow$ bounded. $x_{n_k} \leq d \ d \in S_\Omega \ x_{n_k} \notin (d, \Omega]$
$x_{n_k} \not\to \Omega \Rightarrow x_n \not\to \Omega \lightning$

Now assume $(x_n) \subset X$ $x_n \to x^1, x^2 \in X$ $x^1 \neq x^2$
If $x^1 \in S_\Omega \Rightarrow (x_n)$ bounded $x^2 \neq \Omega, \Omega'$
$x_n \to x^1, x^2$ in $\overline{S}_\Omega \in T_2 \Rightarrow x^1 = x^2$
Similarly if $x^2 \in S_\Omega$
So return to test $\{x^1, x^2\} = \{\Omega, \Omega'\}$
but this gives $\lightning$ by previous lemma
Thus $x^1 = x^2$ and X has unique limit property

Theorem X ordered $\Rightarrow X$ is T_2 in order topology
$(X, \mathcal{A})$ is T_2 $Y \subset X \Rightarrow (Y, \mathcal{A}_Y)$ is T_2
$(X, \mathcal{A})(Y, \mathcal{B})$ $T_2 \Rightarrow (X \times Y, \mathcal{A} \times \mathcal{B})$ is T_2

!! If $\exists x_1, x_2, \ldots, x_n \in A$ $x_n \to x$ $x_n \neq x$
then x is an accumulation point of A
but the converse is false $A_{\text{lim}} \subset A_{\text{acc}}$

Definition limit points $A_{\text{lim}} := \{x \in X : \exists x_n \in A \setminus \{x\} \ x_n \to x\}$ (I do not use like in book)
$A_{\text{lim}} \subsetneq A_{\text{acc}}$ can happen (I will give you ex. later)
This is why be careful you understand well how a "limit point" is meant!
(return to this p.19 and 34)
In "metrizable" spaces like $\mathbb{R}$, $\mathbb{R}^n$ with Enclidean topology, it's okay.

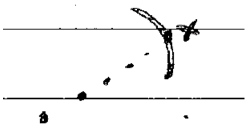


Figure 2 - 19

Theorem $(x_n, y_n) \to (x, y)$ $\Leftrightarrow$ in X $x_n \to x$ in $\mathcal{A}$
and in Y $y_n \to y$ in $\mathcal{B}$ (separation axiom, section 31)

Definition A space $(X, \mathcal{A})$ is regular if (p. 194)
x_1 $A = \overline{A} \not\ni x$ $\exists O_1 \ni x$ $O_i \in \mathcal{A}$
$O_2 \supset A$ $O_1 \cap O_2 = \varnothing$

Figure 2 - 20

Definition A space $(X, \mathcal{A})$ is normal if
$\forall A_1, A_2$ $A_1 = \overline{A_2}$ $A_2 = \overline{A_2}$ $A_1 \cap A_2 = \varnothing$
$\exists O_1, O_2 \in \mathcal{A}$ $O_1 \cap O_2 = \varnothing$ $O_i \supset A_i$

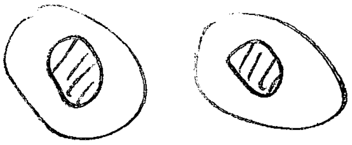


Figure 2 - 21

If points are closed, then normal $\Rightarrow$ regular $\Rightarrow$ T_2
$\Updownarrow$
T_1 but not always otherwise!

So we have a diagram

(! In other books, T_3 = normal, T_4 = regular)

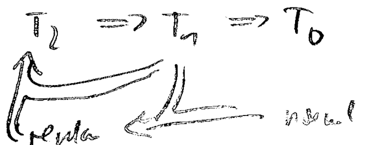


Figure 2 - 22

Definition T_3 is regular Hausdorff (=regular Fréchet)
T_4 is normal Hausdorff (=normal Fréchet)

Theorem order topologies are normal and T_2(and hence anything else)

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① Convergence behavior

X space
$\mathcal{A}$ topology on $X \longmapsto$ convergence behavior of $\mathcal{A}$ $\mathcal{C}(\mathcal{A})$
$\mathcal{A} \subset \mathcal{P}(X)$ $(x_i) \in X^\omega \mapsto \{\text{limit point of } (x_i)\}$
$= \{x \in X : x_i \to x\} \in \mathcal{P}(X)$

$$\mathcal{A} \in \underbrace{\{\text{topology on } X\}}_{\tilde{S}(X)} \subset \mathcal{P}(\mathcal{P}(X))$$

Convergence behavior functor $\tilde{S}(X) \underset{\mathcal{C}}{\rightarrow} \mathcal{F}(X^\omega, \mathcal{P}(X))$

Question Does the convergence behavior determine the topology
i.e. is the convergence behavior functor injective?

Ex. $(X^\omega, \mathcal{A})$ and you prove $f_n \to f$ in $\mathcal{A} \Leftrightarrow f_n \to f$ pointwise
$\Rightarrow \mathcal{A} =$ product topology?
or X metrizable you prove $f_n \to f$ in $\mathcal{A} \Leftrightarrow f_n \rightrightarrows f$ uniform
$\Rightarrow \mathcal{A} =$ uniform topology?

The answer is no! in general
i.e. one cannot identify a topology from its convergence behavior alone

Ex. Consider $X = \overline{S}_\Omega \cup \{\overline{\Omega}\} = \overline{S}_\Omega \cup \{\Omega, \Omega'\}$ Define a topology
$\mathcal{A}$ on X with topology basis
Thus let $\tilde{\mathcal{B}} = \{(a,b) : a < b \ a, b \in \overline{S}_\Omega\} \cup \{[a_0, b) : b \in \overline{S}_\Omega\} \cup \{(a, \Omega] : a \in S_\Omega\}$
$\cup \{[a_0, \Omega]\}$
$\mathcal{B} = \tilde{\mathcal{B}} \cup \{(B \backslash \{\Omega\}) \cup \{\Omega'\} : \Omega \in B \in \tilde{\mathcal{B}}\}$

This is the topology as in the previous example.
$\mathcal{A} = \mathcal{A}(\mathcal{B})$ is not Hausdorff but has unique limit property

Now consider $\mathcal{A}' = \mathcal{A}(\mathcal{B}')$ with $\mathcal{B}' = \mathcal{B} \cup \{\{\Omega\}\}$
(again to check easily $\mathcal{B}'$ is a basis)

$\mathcal{A}'$ is Hausdorff but $\mathcal{A}$ and $\mathcal{A}'$
have same convergence behavior: $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}')$
$x_n \to x$ in $\mathcal{A} \Leftrightarrow x_n \to x$ in $\mathcal{A}'$

Thus the convergence behavior does not define the topology, not even whether it is T_2
$\Rightarrow$ not even up to self-transform fixing convergence behavior

$$\left[\begin{array}{l}
h : X \to X \text{ bijective} \\
\mathcal{A} \mapsto \{h(A) : A \in \mathcal{A}\} = h(\mathcal{A}) \text{ is a topology} \\
\mathcal{F}(\underset{\underset{h}{\curvearrowright}}{X^\omega}, \underset{\underset{h}{\curvearrowright}}{\mathcal{P}(X)}) \quad \mathcal{C}(\mathcal{A}) \mapsto \underset{\underset{\mathcal{C}(h(\mathcal{A}))}{\shortparallel}}{h(\mathcal{C}(\mathcal{A}))} \\
\text{if } \mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}') \quad \exists h : X \to X \text{ bijective} \\
\qquad\qquad\qquad \mathcal{A}' = h(\mathcal{A}) \text{ s.t. } \mathcal{C}(\mathcal{A}) = \mathcal{C}(h(\mathcal{A}))? \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{even this answer is } \underline{\text{no}}!
\end{array}\right]$$

Remark You can consider as well
$X = \overline{S}_\Omega \quad \mathcal{A} = \mathcal{A}(\tilde{\mathcal{B}})$
$\mathcal{A}' = \mathcal{A}(\tilde{\mathcal{B}} \cup \{\{\Omega\}\})$
Then $\mathcal{A} \neq \mathcal{A}'$ but $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}')$
however in this case obviously both $\mathcal{A}, \mathcal{A}'$ are T_2
and I don't know if $\nexists$ self-transformation h on $\overline{S}_\Omega$ with $h(\mathcal{A}) = \mathcal{A}'$
likely not, but it requires some argument to prove
while in our way we get this + T_2 - independence readily
for example, $\mathcal{A} \ni \{x\}$ one element set
$\Leftrightarrow$ x has no immediate predecessor
or $x = a_0$
(x has always an immediate successor or $x = \Omega$
but $\mathcal{A}$ contains uncountably many 1-element sets (old exercise)
so then one cannot argue with number of of 1-element sets that $\nexists h$, etc.. . .)

② Limit points

Exercise let X be a topological space
$x_1 \in \mathcal{F}(\mathbb{Z}_+, X)$ a sequence
let $h : \mathbb{Z}_+ \to \mathbb{Z}_+$ bijective
then $x_n \to x$ in $X \Leftrightarrow x_{h(n)} \to x$ in X
This means ordering a sequence is not relevant for its limit(s)
Then the limit points of a sequence $\{x \in X : x_n \to x\}$
can in fact be defined on the set$\{x_n\}$

Definition X topological space $A \subset X$ $|A| = \omega$
define $A^{\mathrm{lim}} = \{x \in X : \exists h : \mathbb{Z}_+ \to A \text{ bijective } h(n) \xrightarrow[n\to\infty]{} x\}$

Note, however, that $A^{\mathrm{lim}} \neq A_{\mathrm{lim}}$
the set of limit points of A as a set, as defined in the closure section

In fact, $A_{\mathrm{lim}} = \bigcup_{A' \subset A, |A'| = \omega} (A')^{\mathrm{lim}}$ (if X is T_1)
also for any subset $A \subset X$ (not necessarily of cardinality ω)

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20. The metric Topology

most important and fundamental source of topology

Definition X set $d(x, y) > 0$ $x \neq y$ distance if
$d(x, x) = 0 \quad d(x, y) > 0$ $x \neq y$
$d(x, y) = d(y, x)$
$d(x, y) + d(y, z) \geq d(x, z)$
$d(x, y)$ distance between x and y

Definition $B_\epsilon(x) = \{y \in x : d(x,y) < \epsilon\}$ $\epsilon > 0, x \in X$
(open) ball ϵ-ball centered at x

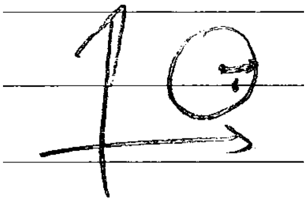


Figure 2 - 23

Definition The metric topology $\mathcal{A}_d$ on (X,d) induced by d is the one with basis $\{B_\epsilon(x) : x \in X,\ \epsilon > 0\}$

To check: balls form a basis

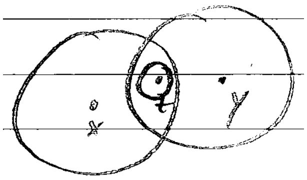


Figure 2 - 24

B1) $X \supset \bigcup_{x \in X} B_\epsilon(x) \supset \bigcup_{x \in X} \{x\} = X$
B2) $\exists \epsilon(x) \cap B_{\epsilon'}(y)$
$B_{\epsilon''}(z) \subset B_\epsilon(x) \cap B_{\epsilon'}(y)$
for $\epsilon'' = \min(\epsilon - d(x,z), \epsilon' - d(y,z)) > 0$

Definition $(X, \mathcal{A})$ metrizable $\Leftrightarrow$ $\exists d$ on X
with $\mathcal{A} = \mathcal{A}_d$ induced by d

Ex. $X, \delta(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
discrete distance $B_1(x) = \{x\}$ so
it induces the discrete topology $\mathcal{A} = \mathcal{P}(X)$
↑ explains the name
$A \subset X$ every $a \in A$ is discrete point, every set is discrete
so the discrete topology is metrizable

Ex. Standard metric on $\mathbb{R}$ $d(x,y) = |x - y|$
induces the usual (Euclidean) topology
because $(a,b) = B_\epsilon(x)$ for $x = \dfrac{a+b}{2}$ $\epsilon = \dfrac{b-a}{2}$
Euclidean topology is metrizable
$\Rightarrow$!non-uniquely $\hat{d}(x,y) = 2|x - y|$ induces the same topology.

Ex. $\mathbb{R}$ finite complement topology not metrizable

if were $\bigcap_{n=1}^{\infty} \underbrace{B_{y_n}(x)}_{\text{open}} = \{x\}$

$$\underbrace{\mathbb{R}\backslash\{x\}}_{\text{uncountable}} = \underbrace{\bigcup_{n=1}^{\infty}(\underbrace{\mathbb{R}\backslash B_{y_n}(x)}_{\text{finite}})}_{\text{countable}}$$

Definition V Vector Space over $\mathbb{R}$ or $\mathbb{C}$

a norm $||.|| : V \to [0,\infty)$

satisfies $||\mathbf{v}|| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}_V$

$||\lambda\mathbf{v}|| = |\lambda| \cdot ||\mathbf{v}|| \quad \lambda \in \mathbb{R}(\mathbb{C})\ \mathbf{v} \in V$

$||\mathbf{v}|| + ||\mathbf{w}|| \geq ||\mathbf{v}+\mathbf{w}||$ e.g. $||\mathbf{v}|| = \sqrt{<\mathbf{v},\mathbf{v}>}$ if $\exists <,>$

$||.||$ induces a metric by $d(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}||$

{inner product space}$\subset${normed space}$\subset${metric space}$\subset${topological space} all $\subsetneq$

we see here later $\uparrow$

Ex. $\mathbb{R}^n$ with Euclidean norm

$||\mathbf{x}|| = ||(x_1,\ldots,x_n)|| = \sqrt{x_1^2+\cdots+x_n^2}$

induces the product topology on $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$

basis are $\{(a_1,b_1)\times\cdots\times(a_n,b_n) : a_i < b_i\}$

Ex. The Hölder p-norm

$||\mathbf{x}||_p = \sqrt[p]{|x_1|^p+\cdots+|x_n|^p}$ for $p \in [1,\infty)$

are not multiples of $||.|| = ||.||_2$ for $p \neq 2$

but all induce the same topology on $\mathbb{R}^n$

Definition (X,d) metric space $A \subset X$ $x \in X$ $\text{dist}(x,A) = \inf\{d(x,a) : a \in A\}$

Lemma $\text{dist}(x,A) = 0 \Leftrightarrow x \in \overline{A}$

Definition (X,d) metric space $A \subset X$

diam $A = \sup\{d(a,a') : a,a' \in A\}$

Ex. diam $B_\epsilon(x) \leq 2\epsilon$ (not always "=")

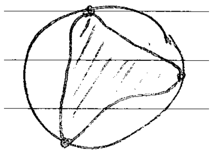


Figure 2 - 25

diam can be ∞ if d is unbounded

Ex. diam $(\mathbb{Z}_+) = \infty$ $\quad \mathbb{Z}_+ \subset \mathbb{R}$ with Euclidean metric

Research Problem $A \subset \mathbb{R}^2$ diam $(A) \leq 1$ $A \subset \overline{B}_{1/\sqrt{3}}(x)$ for some x?

$1/\sqrt{3}$ smallest possible?

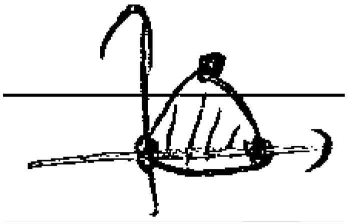


Figure 2 - 26

sometimes it is useful to bound d

Theorem Let (X, d) metric space
Define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d} = \max(d, 1)$
standard bounded metric

$\overline{d}$ induces $\mathcal{A}_{\overline{d}} = \mathcal{A}_d$, same topology

Lemma $(X, d), (X, d')$ $\mathcal{A}_{d'} \supset \mathcal{A}_d$
$\Leftrightarrow \forall x \in X\ \epsilon > 0\ \exists \delta \quad B_{d',\delta}(x) \subset B_{d,\epsilon}(x)$

Lemma $A \subset (X, d)$ $\mathcal{A}_{d|A\times A}$ is the relative topology of d of A

if $\mathcal{A}$ is metrizable $\Rightarrow \mathcal{A}_A$ is metrizable

Theorem (X, d) topological space
$x_n \to x \Leftrightarrow \forall \epsilon > 0\ \exists N\ \forall n > N\ d(x_n, x) < \epsilon\ x_n \in B_\epsilon(x)$

Theorem metric spaces are Hausdorff and normal (so anything else in chart p.14)
pf. mostly the idea for order topologies (exercise)

Theorem (X, d) metric $x \in \overline{A} \Leftrightarrow \exists x_n \in A\ x_n \to x$

Sequence Lemma
i.e. trouble at p.14 does not occur for metrizable spaces

Remark: if X, Y metrizable with d_X,d_Y, then $X \times Y$ is also metrizable (e.g.) with $d(x_1 \times y_1, x_2 \times y_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

18. Continuous Maps

Definition $(X, \mathcal{A})$ $(Y, \mathcal{B})$
$f : X \to Y$ topological space continuous
if and only if $\forall O \in \mathcal{B}$ open in Y $\quad f^{-1}(O)$ open in X
continuous relative to the topologies $\mathcal{A} \subset \mathcal{B}$

Ex. $(X, \mathcal{A})$ $(X, \mathcal{B})$
$\mathcal{A}$ finer than $\mathcal{B} \Leftrightarrow \mathrm{id}_X : (X, \mathcal{A}) \to (X, \mathcal{B})$ is continuous

Remark (book p.90) if $\forall O \in \mathcal{A}\ f(O) \in \mathcal{B}$, call f open

Lemma Let $(X, \mathcal{A})$ have basis $\mathcal{A}'$, $(Y, \mathcal{B})$ have basis $\mathcal{B}'$

$$f \text{ continuous } \Leftrightarrow \forall B' \in \mathcal{B}'\ f^{-1}(B') \in \mathcal{A}$$
$$\Leftrightarrow \forall B' \in \mathcal{B}\ \forall x \in f^{-1}(B') \quad \exists A' \in \mathcal{A}'\ x \in A' \subset f^{-1}(B')$$

("$\epsilon - \delta$" and sequence condition)

Lemma if (X, d) (X, d') are metric spaces can take balls as bases

$$f : X \to Y \text{ continuous } \Leftrightarrow \forall \epsilon > 0\ \exists \delta > 0 \quad B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$$
$$\Leftrightarrow \forall x_n \to x \text{ in } X \quad f(x_n) \to f(x) \text{ in } Y$$

Theorem X, Y topological spaces $f : X \to Y$ The following are equivalent

1) f continuous
2) $\forall A \subset X \quad f(\overline{A}) \subset \overline{f(A)}$
3) $\forall B \subset Y$ closed $f^{-1}(B) \subset X$ closed
4) $\forall x \in X \quad \forall U \ni f(x)$ open $\exists V \ni x$ open
   $V \subset f^{-1}(U)$ ($\Leftrightarrow f(V) \subset U$)

Definition f homeomorphism if
$f : X \to Y$ bijective, f, f^{-1} continuous
$(X, \mathcal{A})$ $(Y, \mathcal{B})$

$\mathcal{A} = \{f^{-1}(B) : B \in \mathcal{B}\}$ $\mathcal{B}$ determines $\mathcal{A}$

$X \simeq Y$ topologically equivalent
if $\exists f : X \to Y$ homeomorphism

Definition $f : X \hookrightarrow Y$ continuous, injective
if $f^{-1} : f(X) \to X$ continuous,
call f a (topological) embedding

Ex. $f : [0, 2\pi) \to S^1 = \{(x, y) : x^2 + y^2 = 1\}$ $f(t) = (\cos t, \sin t)$
with relative topology to $\mathbb{R}^2$
is bijective and continuous
but its inverse is not continuous $\Leftrightarrow$ f is not open
$[0, \alpha)$ $T = f([0, \alpha)) \subset S^1$ is not open $1 \in S^1$ but $\forall \epsilon > 0$ $S^1 \cap B_\epsilon(1) \notin T$
$\cap$
$[0, 2\pi)$ open

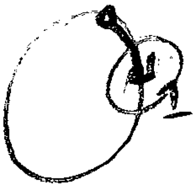


Figure 2 - 27

Constructing Continuous Functions

see p.105, 106 in the book

Pasting lemma

$X = A \cup B$ A, B closed in X
$f : X \to Y$ $f|_A : A \to Y$
$f|_B : B \to Y$ continuous $\Rightarrow f : X \to Y$ continuous

Maps into product

Theorem $f : A \to X \times Y$(with product topology) is continuous $\Leftrightarrow$
coordinate functions $f_1 : A \to X \quad f_1 = \text{pr}_X \circ f$
$f_2 : A \to Y \quad f_2 = \text{pr}_Y \circ f$
are continuous. ($\text{pr}_X(x, y) = x$, $\text{pr}_Y(x, y) = y$)

19. Product Topology II

Motivation: $X_1 \times \dots \ X_n$ with topology $\mathcal{A}_1 \dots \mathcal{A}_n$

defined product topology by
(1) subbasis / for $n = 2$ but finite n is same story

$$\bigcup_{k=1}^{n} \{\text{pr}_k^{-1}(U) : U \subset X_k \text{ open}\}$$

$$U = X_1 \times \dots X_{k-1} \times U \times X_k \times \dots \times X_n$$

or

(2) basis

$$\{U_1 \times \dots U_k : U_i \subset X_i \text{ open}\}$$

gives (same) product topology
But what if ∞ product? There's a difference!
First let's define general tuple indexed by an arbitrary set.

Convention write $\mathcal{F}(X, Y)$ for $\{f : X \to Y\}$
Recall $\{f : \mathbb{Z}_+ \to X\} \simeq X^\omega = \{(a_1, a_2, \dots) : a_i \in X\}$
$(= \mathcal{F}(\mathbb{Z}_+, X) \uparrow f(n) = a_n)$
$= \prod_{i=1}^{\infty} X$

$$\prod_{i=1}^{\infty} X_i = \{f : \mathbb{Z}_+ \to \bigcup X_i : f(n) \in X_n \ \forall n\}$$

For J index set, X set, define J-tuple of elements in X to be a function $x : J \to X$

$$x = (x_\alpha)_{\alpha \in J} \qquad x(\alpha) = x_\alpha$$

$$X^J = \{x : \ \ x(\alpha) \in X \ \forall \alpha \in J \ \} \simeq \{f : J \to X\}$$

$\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$
Definition $\prod_{\alpha \in J} A_\alpha = \{f : J \to \bigcup_{\alpha \in J} A_\alpha : f(\alpha) \in A_\alpha \ \ \forall \alpha \in J\}$
J - indexed product

Definition $\{A_\alpha\}_{\alpha \in J} \quad (A_\alpha, \mathcal{A}_\alpha)$ topological space
the box topology on $A = \prod_{\alpha \in J} A_\alpha$ is defined by the basis
$\{U = \prod_{\alpha \in J} U_\alpha : U_\alpha \in \mathcal{A}_\alpha \ \ \forall \alpha \in J\}$
"U is a box" generalization of (2) above

Definition $(A_\alpha, \mathcal{A}_\alpha)$ topological space
the product topology on $A = \prod A_\alpha$ is defined by the subbasis
$\bigcup_{\alpha \in J} \bigcup_{U_\alpha \in \mathcal{A}_\alpha} \mathrm{pr}_\alpha^{-1}(U_\alpha)$
with $\mathrm{pr}_\alpha : A \to A_\alpha$ being the projection on the
α-th coordinate $\mathrm{pr}_\alpha((a_\beta)_{\beta \in J}) = a_\alpha$ generalization of (1)

The product topology will be assumed by default!

Obviously, box topology $\supset$ product topology.
but box topology is quite strong and often good for counterexamples, while product topology is used in many theorems.

For metric spaces (A_α, d_α), there is one more important topology on $\prod A_\alpha$, the uniform topology. It has basis being boxes of “equal length”

$$\mathcal{B} = \{\prod_{\alpha \in J} B_\epsilon(x_\alpha) : x_\alpha \in A_\alpha, \epsilon > 0\}$$

ϵ does not depend on α!

Theorem the uniform topology is induced by the uniform metric
$d((a_\alpha)_{\alpha \in J}, (b_\alpha)_{\alpha \in J}) = \sup\{\overline{d}_\alpha(a_\alpha, b_\alpha) : \alpha \in J\}$
↑ bounded metric

Convergence

Notation write $\mathcal{F}(X,Y)_{\text{prod}}$ $\mathcal{F}(X,Y)_{\text{box}}$ $\mathcal{F}(X,Y)_{\text{uni}}$

Theorem Convergence in the product topology is pointwise convergence

Definition $f_n \to f :\Leftrightarrow \forall \alpha \in J \quad f_n(\alpha) \to f(\alpha)$ in A_α
$\forall \alpha \in J \quad \forall O \in \mathcal{A}_\alpha \ O \ni f(\alpha) \quad \exists N \ \forall n \geq N \ f_n(\alpha) \in O$

Th./Def. Convergence in the uniform topology is the uniform convergence
$f_n \rightrightarrows f :\Leftrightarrow \forall \epsilon > 0 \ \exists N \ \forall n \geq N \ \forall \alpha \in J \ d_\alpha(f_n(\alpha), f(\alpha)) < \epsilon$
remark: $f \to f \quad \forall \alpha \in J \ \forall \epsilon > 0 \ \exists N \ \forall n \geq N \ d_\alpha(f_n(\alpha), f(\alpha)) < \epsilon$

What is convergence in the box topology?

Ex. Consider $\{f : \mathbb{R} \to \mathbb{R}\}$ with box topology. When $f_n \to 0$?
assume $f_{n_1}(x_1) \neq 0$ choose $U_{x_1}(-f_{n_1}(x_1), f_{n_1}(x_1)) \quad f_1 \notin U = \prod_{x \in \mathbb{R}} U_x$
assume $\exists x_2 \neq x_1 \quad n_2 > n_1 \quad f_{n_2}(x_2) \neq 0 \quad U_{x_2}(-f_2(x_2), f_2(x_2)) \quad f_2 \notin U = \prod_{x \in \mathbb{R}} U_x$
$\rightsquigarrow$ find $(f_{n_1}, f_{n_2}, \dots) \notin U \ni 0$ open. $\to f_n \not\to 0$
etc. when does this fail?
$f_n \to 0 \Leftrightarrow \exists N \ S_N := \bigcup_{n \geq N} \{x : f_n(x) \neq 0\}$ finite
and $\forall x \in S_N \quad f_n(x) \to 0$
$(\Leftrightarrow f|_{S_N} \to 0|_{S_N} \Leftrightarrow f_n|_{S_N} \rightrightarrows 0|_{S_N})$

Ex. $f_n(x) = \begin{cases} 0 & x \neq n \\ 1/n & x = n \end{cases}$ (so box convergence $\Rightarrow$ pointwise and uniform convergence)

of course $f_n \to 0$ and even $f_n \rightrightarrows 0$
also $|\{x : f_n(x) \neq 0\}| = 1 < \infty \forall n$
but $\bigcup\limits_{n \geq N} \{x : f_n(x) \neq 0\} = \mathbb{Z}_+ \cap [N, \infty)$ infinite
so $f_n \not\to 0$ (and not to any other limit) in box topology

The box topology is easily seen to be Hausdorff and I can prove regular. It is not known about normal see Ex.5 p.203

Theorem In product or box topology of $\mathcal{F}(J, X)$
(and hence in the uniform topology as well, if X_α metric)
for $A_\alpha \subset X_\alpha$ $\overline{\prod\limits_{\alpha \in J} A_\alpha} = \prod\limits_{\alpha \in J} \overline{A_\alpha}$

Ex. Consider $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ in box topology. $\mathcal{F} = \mathcal{F}(\mathbb{R}, \mathbb{R})_{\text{box}}$
Let $A = \prod\limits_{\alpha \in \mathbb{R}} (0, 1) = \{f : \mathbb{R} \to \mathbb{R} : \text{image}(\mathbb{R}) \subset (0, 1)\}$
Then $\overline{A} = \prod\limits_{\alpha \in \mathbb{R}} [0, 1] = \{f : \mathbb{R} \to \mathbb{R} | \text{image}(\mathbb{R}) \subset [0, 1]\}$
but consider $A_{\text{lim}} = \{f \in \mathcal{F} : \exists f_n \in A \backslash \{f\} : f_n \to f\}$

$$A_{\text{lim}} = \bigcup_{\substack{S \subset \mathbb{R} \\ |S| < \infty}} \prod_{\alpha \in \mathbb{R}} \left\{ \begin{matrix} [0, 1) & \alpha \notin S \\ [0, 1] & \alpha \in S \end{matrix} \right\}$$
$$= \{f : \mathbb{R} \to \mathbb{R} \mid \text{image}(f) \subset [0, 1],\ \exists S \subset \mathbb{R}\ |S| < \infty\ \text{image}(f|_{\mathbb{R} \backslash S}) \subset (0, 1)\}$$

E.g. $0 \in A_{\text{acc}}$ $0 \notin A_{\text{lim}}$ (0 =zero function)
$A_{\text{lim}} \subsetneq A_{\text{acc}} \Rightarrow$ sequence lemma fails $\Rightarrow$ box topology not metrizable
$\Rightarrow$ ($\mathcal{A}' \supset \mathcal{A}$ $\mathcal{A}$ metirzable$\not\Rightarrow$ $\mathcal{A}'$ metrizable)
$\uparrow$ box $\uparrow$ uniform while T_i $i \leq 2$

It took me some to give an example of $A \subset X$ with

$$\overline{A} \backslash A \neq \varnothing \text{ but } A_{\text{lim}} = \varnothing, \text{ i.e.}$$

A not closed ($\Rightarrow$ has accumulation point $A_{\text{acc}} \neq \varnothing$) but no converging sequence(except constant)

Ex. (may skip, not very good)

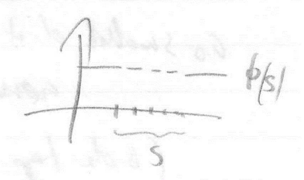
Consider $X = \{f : \mathbb{R} \to \mathbb{R}_{\text{discr}}\}$ with range $\mathbb{R}$ with discrete topology
and pointwise convergence topology $X = \mathcal{F}(\mathbb{R}, \mathbb{R}_{\text{discr}})_{\text{prod}}$ (product discrete topology)
i.e. $f_n \to f \Leftrightarrow \forall x \in \mathbb{R} \quad \exists N \quad \forall n \geq N \quad f_n(x) = f(x)$
the basis of the topology

$$\{U_{S,y_1,\ldots,y_n} = \prod_{x\in\mathbb{R}} \left\{ \begin{array}{ll} \{y_i\} & x = x_i \\ \mathbb{R} & x \neq x_1, \ldots, x_n \end{array} \right\}$$

$$: S = \{x_1, \ldots, x_n\} \subset \mathbb{R} \quad \text{finite } (x_i \neq x_j), \qquad y_1, \ldots, y_n \in \mathbb{R}\}$$

Fix a bijection $\Phi : \{\text{finite subsets of } \mathbb{R}\} \to \mathbb{R}\backslash\{0\}$
Consider the family

$$f_S(x) = \left\{ \begin{array}{ll} 0 & x \in S \\ \Phi(S) & x \neq S \end{array} \right\}$$



constant function except for finite zero set

Figure 2 - 28

$A = \{f_S : S \subset \mathbb{R} \text{ finite}\} \subset X$
$\downarrow$ function $\quad (f(x) = 0 \quad \forall x \in \mathbb{R})$
$0 \in \overline{A}\backslash A \qquad A$ has accumulation point 0

If $0 \in U$ open $\quad U$ basis element
$U = U_{S,\underbrace{0,\ldots,0}_{y_i=0}}$ for some $S \subset \mathbb{R} \quad |S| < \infty$
$f_S \in U \cap A \neq \varnothing \Rightarrow 0 \in \overline{A}$, obviously $0 \notin A$
Now take sequence $f_{S_1}, f_{S_2}, \ldots$ in $A \quad f_{S_i} \neq f_{S_j} \; i \neq j$
$\bigcup_{i=1}^{\infty} S_i$ is countable, take $x_0 \in \mathbb{R}\backslash\bigcup_{i=1}^{\infty} S_i$
$f_{S_n}(x_0) = \Phi(S_n) \quad f_{S_n}(x_0) \neq f_{S_m}(x_0) \quad \forall n, m \in \mathbb{Z}_+ \; n \neq m$
$f_{S_n}(x_0)$ does not converge in discrete topology
$\Rightarrow f_{S_n} \not\to f$
So A contains no converging sequence! (except eventually constant)
($\Rightarrow$ X not metrizable etc.) $\quad A_{\text{acc}} \ni 0$, $A_{\text{lim}} = \varnothing \quad$ End Ex.

Note: $A_{\text{lim}} = \varnothing$ means A has no limit point in X, not in A.
A has no limit point in A just means A is discrete,
and I can find you discrete non-closed sets in
simple spaces like $A = \{\frac{1}{n} : n \in \mathbb{Z}_+\} \subset \mathbb{R} = X$

Order topologies II

Theorem $(X, <)$ ordered set $\Rightarrow$ order topology is normal

Ex. $\mathbb{R} \times \mathbb{R}$ with dictionary order is metrizable
$\simeq (0,1) \times (0,1)$ HW Ex. 2.20.2 p. 124
ordered square $I_0^2 = [0,1] \times [0,1]$is not, proof later! see pg. 30
in particular, restricting the order topology of
$\mathbb{R} \times \mathbb{R}$ to $[0,1] \times [0,1]$ does not give the
same topology as the topology from restricting
dictionary order on $[0,1] \times [0,1]$ see pg. 10

(My) *pf.* Let $(X, <)$ be ordered.
$A, B \subset X$closed disjoint fixed.

idea. $\forall a \in A$ construct $(x_a, y_a) \ni a \quad (x_a, y_a) \cap B = \varnothing$
$\forall b \in B \qquad (\overline{x_b}, \overline{y_b}) \ni b \quad (\overline{x_b}, \overline{y_b}) \cap A = \varnothing$
$\forall a, b \quad (x_a, y_{,a}) \cap (\overline{x_b}, \overline{y_b}) = \varnothing$ $(*)$ see pg. 27

Let then $O_1 = \bigcup\limits_{a \in A} (x_a, y_a) \quad O_2 = \bigcup\limits_{b \in B} (\overline{x_b}, \overline{y_b})$
$O_1 \supset A \quad O_2 \supset B \quad O_1 \cap O_2 = \varnothing$
Need to define $x_a, y_a, \overline{x_b}, \overline{y_b}$
First, need to take care of blank intervals

Definition a) (recall) $I \subset X$ convex if
$\forall a, b \in I \quad a < b \quad \forall x \in X \quad a < x < b \Rightarrow x \in I$

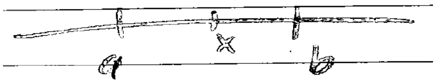


Figure 2 - 29

b) $I \subset X$ is called blank interval(BI) if
I convex and $I \cap (A \cup B) = \varnothing$

c) $I \subset X$ is maximal b.i.(MBI) if
I is BI $\quad \forall$BI $\quad I' \supset I \Rightarrow I' = I$

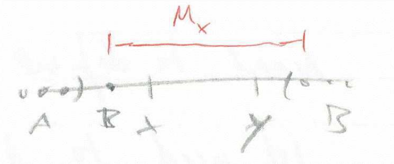
Lemma each BI $I \subset!$ one MBI $\tilde{I} = M_I$

pf. $\exists \max$ BI $\quad \tilde{I} \supset I$ by maximum principle
unique because if $\tilde{I}, \tilde{I}' \supset I \quad \hat{I} = \tilde{I} \cup \tilde{I}' \supset I$
MBI also defines BI $\supset I$

by max $\begin{matrix} \hat{I} \supset \tilde{I} & \tilde{I} \max \Rightarrow \hat{I} = \tilde{I} \\ \hat{I} \supset \tilde{I}' & \tilde{I}' \max \Rightarrow \hat{I} = \tilde{I}' \end{matrix} \Rightarrow \tilde{I} = \tilde{I}'$

In particular, $\forall x \notin A \cup B \quad \{x\}$ is BI
$\exists!$ MBI $\supset \{x\}$ write it M_x

Lemma $\forall x, y \in X \quad [x,y]$ BI $\Leftrightarrow M_x = M_y$



pf. $\Leftarrow \quad M_x = M_y \ni \{x, y\}$ BI

Figure 2 - 30

$[x,y] \subset M_x \Rightarrow [x,y] \cap (A \cup B) = \varnothing \Rightarrow [x,y]$ BI

$\Rightarrow \quad x \notin A \cup B \quad$ Let I BI $\quad I \ni x$

$I \subset$ MBI M_I

$x \in$ MBI M_I

by uniqueness $\quad M_I = M_x$

now $x, y \in [x,y]$ BI; by ↑ above argument

$M_{[x,y]} = M_x = M_y$

Convention $(X,<)$For each MBI $\Delta \subset X$ fix an element, $\alpha_\Delta \in \Delta$.(AC!)
Return to definition of y_a $\qquad$ (namely $M_{\alpha_\Delta} = \Delta$)

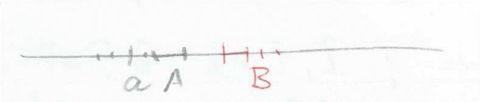


Figure 2 - 31

Assume $a \in A$, So $a \notin B = \overline{B}$

if $a = \max X$ set $y_a = \infty$ $(y_\alpha = \infty$(formally) with $(x_\alpha, y_\alpha) = (x_\alpha, \infty) = (x_\alpha, a]$)

otherwise $\exists a'' > a \quad a''$ is lower bound for $B \cap [a, \infty)$ $B \cap (a, a'') = \varnothing$

[0] a'' is immediate successor of a set $y_a = a''$

Otherwise, $\exists a' \in (a, a'')$ $\quad$ (need to avoid $a' \in B$)

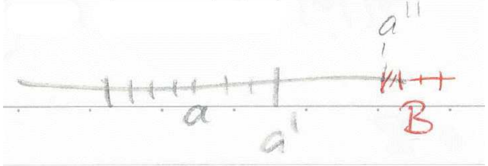


Figure 2 - 32

[1] if $a' \in A$ set $y_a = a'$

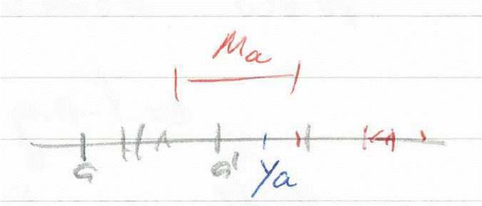


Figure 2 - 33

[2] if $a' \notin A \Rightarrow a' \notin A \cup B \quad$ set $y_a = \alpha_{M_{a'}}$

Then y_a has the following properties

1) $y_a = \infty$ only if $a = \max X$
assume $a \neq \max X$ below

2) $y_a > a$ and $\big([a, y_a] \cap B = \varnothing$ or $y_a = \text{succ}(a)\big)$
$\underline{pf}$.if $y_a \leq a$, then need case [2] and
$\overline{M_{y_a} = M_{\alpha_{M_{a'}}}} = M_{a'} (!)$
$\Rightarrow [y_a, a']$ blank $a \in [y_a, a']$ $a \notin A \lightning$

3) if $y_a \notin A$ then by (!) we have $y_a = \alpha_{M_{y_a}}$ or $y_a = \text{succ}(a)$ is the immed. succ. of a

Similarly $\overline{y_b}$ for $b \in B$
$\overline{x_b}$ should be defined so that

1) $\overline{x_b} = -\infty$ only if $b = \min X$
assume $b \neq \min X$

2) $\overline{x_b} < b$ and $\big([\overline{x_b}, b] \cap A = \varnothing$ or $\overline{x_b} = \text{pred}(b)\big)$

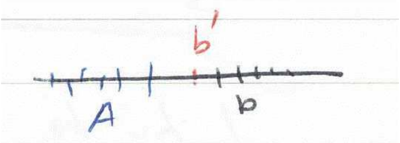


Figure 2 - 34

3) if $\overline{x_b} \notin B$ then $\overline{x_b} = \alpha_{M_{\overline{x_b}}}$ or $\overline{x_b} = \text{pred}(b)$ is the immed. pred. of b

Similarly x_a. So now $x_a, y_a, \overline{x_b}, \overline{y_b}$ defined.
Now assume $a \in A$ $b \in B$
Without loss of generality, $a < b$ $\quad \Rightarrow (y_a \neq \infty$ $\overline{x_b} \neq -\infty)$
assume $(x_a, y_a) \cap (\overline{x_b}, \overline{y_b}) \neq \varnothing$ (**); to prove $(*)$ at page 25 by $\lightning$
So $\overline{x_b} < y_a$

if $y_a = \text{succ}(a)$, then $(x_a, y_a) = (x_a, a]$, so $(**) \Rightarrow \overline{x_b} \leq a$. But $b > a$, so $a \in A \cap [\overline{x_b}, b]$ $\lightning$ to 2)
Similarly $\overline{x_b} \neq \text{pred}(b)$.

$$\begin{tikzcd}
\begin{array}{l} [\overline{x_b}, b] \cap A = \varnothing \\ [a, y_a] \cap B = \varnothing \end{array} & {[\overline{x_b}, y_a] \cap (A \cup B) = \varnothing} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
M_{y_a} = M_{\overline{x_b}} \arrow[dr, Rightarrow] & \begin{array}{l} y_a \notin A \overset{3)}{\Longrightarrow} y_a = \alpha_{M_{y_a}} \\ \overline{x_b} \notin B \overset{3)}{\Longrightarrow} \overline{x_b} = \alpha_{M_{\overline{x_b}}} \end{array} \arrow[d, Rightarrow] \\
 & y_a = \overline{x_b} \lightning
\end{tikzcd}$$

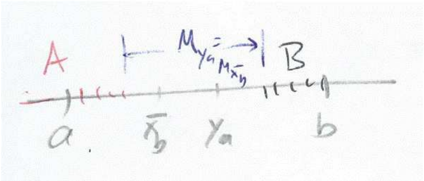


Figure 2 - 35

$\Rightarrow (*)$ at page 25 $\Rightarrow \blacksquare$

Chapter 3. Compactness, Connectedness, Completeness

26. Compact Spaces

I try to discuss this first exclusively for metric spaces and say later how/what to generalize to topological spaces.

Definition $x_n \subset (X,d)$ sequence, x_{n_m} is a subsequence if
$n_m > n_{m-1} \quad n_m \in \mathbb{Z}_+$

Definition (sequence) compactness / limit point compactness (see warning(!!Def.) page. 30)
(X,d) is called compact if
$\forall (x_n) \subset X$ sequence, $\exists (x_{n_m}) \subset (x_n)$ subsequence
converging in $X \quad \exists x \in X : x_{n_m} \to x$ limit point

Remark (X,d) compact $\Rightarrow \operatorname{diam}(X) < \infty$
and must be attained
$\exists x, y \in X \quad d(x,y) = \operatorname{diam}(X)$

ex. $\mathbb{R}$ is not compact
$(0,1)$ is not compact (with Euclidean distance)

Ex. $\mathbb{R}, \overline{d} = \min(d,1)$ bounded, is not compact

Ex. $\mathbb{R}$ with discrete distance is bounded and attains diameter but not compact
(every converging sequence must be eventually constant)

But [0,1] is compact (with Euclidean distance) prove now!

Lemma $x_n \subset \mathbb{R}$

increasing	$x_n \geq x_{n-1}$
strictly increasing	$x_n > x_{n-1}$
decreasing	$x_n \leq x_{n-1}$
strictly decreasing	$x_n < x_{n-1}$
monotonous	= increasing or decreasing

Lemma x_n increasing and bounded (above)
$x_n \to \sup\{x_n\}$
similarly decreasing

Lemma (x_n) bounded $\Rightarrow \exists (x_{n_m})$ monotonous

pf. Assume $\nexists (x_{n_m})$ increasing will show lemma
by finding one decreasing sequence.
$n_0 = 0, k = 0$
Let $x = \sup\{x_n : n > n_k\}$
if infinitely many $x_n = x$ $\quad \exists$ constant subsequence $\Rightarrow$ increasing $\lightning$
if exists no $x_n = x$
$\exists n_1 \quad x_{n_1} > x - 1 \quad x_{n_1} < x$
$\max\{x_1, \dots, x_{n_1}\} < x$
$\exists n_2 > n_1 \quad x_{n_2} > x_{n_1} \quad x_{n_2} < x$
x_{n_i} increasing $\lightning$
So $\exists$ finitely many $x_n = x$
Let $n_{k+1} = \max\{k : x_k = x\} \quad x_{n'} < x_{n_{k+1}} (= x) \quad n' > n_{k+1}$
x_{n_k} decreasing

Theorem (Bolzano-Weierstrass)
Every bounded sequence in $\mathbb{R} \supset$ converging subseqence

Ex. $[0,1] \quad (x_n) \subset [0,1]$
$x_n \subset \mathbb{R}$ bounded $\Rightarrow x_{n_m} \to x$
$[0,1]$ closed $\Rightarrow x \in [0,1]$
$x_{n_m} \to x \in [0,1]$ in relative topology
$\Rightarrow [0,1]$ compact

Theorem $(X_1, d_1), (X_2, d_2)$ compact
$\Rightarrow (X_1 \times X_2, d_1 + d_2)$ compact
$\uparrow$ or any d inducing product topology $\sqrt{d_1^2 + d_2^2}$ ...

Theorem (X, d) compact $A \subset X$ compact (in relative topology)
$\Leftrightarrow A = \overline{A} \qquad d|_{A \times A}$

Theorem (X, d) any metric space $\quad A \subset X$ compact
$\Rightarrow A$ closed and bounded

Theorem (Heine-Borel) $X = \mathbb{R}^n$ with Euclidean metric

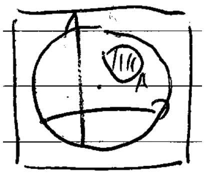


Figure 2 - 36

A closed and bounded $\Leftrightarrow$ compact $\qquad A \subset B \subset I \times I'$

Ex. $[0,1]^\omega \subset \mathbb{R}^\omega$ with uniform metric closed and bounded
but $\exists f_n = \left\{ \begin{array}{ll} 1 & x = n \\ 0 & x \neq n \end{array} \right\} \quad \nexists g : f_{n_m} \rightrightarrows g$ uniformly

Thoerem $f:(X,d)\to\mathbb{R}$ continuous X compact
f bounded and f attains maximum value

pf. if $\exists f(x_n)\to\infty \quad x_n\in X$
$\exists x_{n_m}\subset x_n$ converging $x_{n_m}\to x$
f continuous $f(x_{n_m})\to f(x)\in\mathbb{R}$↯
so assume $f(x_n)$ bounded
$\forall\epsilon>0 \quad \exists x_\epsilon\in x \quad f(x_\epsilon)>\sup(f(x))-\epsilon$
$(x_{1/n})$ has converging subsequence $x_{n_m}\to x\in X$
$f(x_{n_m})\to f(x)$
↳ $\sup f \qquad f(x)=\sup f=\max f$

Corollary (X,d) compact $\Rightarrow$ X attains diam $<\infty$

pf. $d:\underbrace{X\times X}_{\text{compact}}\to\mathbb{R}$ continuous

↓ I use; not to confuse with my use of "limit point"(pg. 28)

!! Definition (X,d) is acc. point compact
$\Leftrightarrow \forall S\subset X \quad |S|=\infty \quad S$ has acc. point $\quad A_{\text{acc}}\neq\emptyset$

Lemma (X,d) acc. point compact $\Leftrightarrow$ sequentially compact

pf. $\Leftarrow$ $|S|=\infty \quad \exists s_1,s_2,\dots\in S \quad s_i\neq s_j$
⇘
$\exists s_{n_m}\to X$ at most one $S_{n_m}=X$
X acc. point of S
$\Rightarrow$ $(x_n)\subset X$ sequence, if set $\{x_n\}$ finite
then $\exists$ finitely many equal $x_{n_m}\to$ done
so $|\{x_n\}|=\infty$
$\Rightarrow$ it has accumulation point x
choose $x_{n_m}\in B_{1/m}(x) \quad x_{n_m}\to x$

Theorem (see pg. 25) The ordered square $[0,1]^2$ with the
order topology of the lexicographical order is not metrizable
(but it is T_4 like any order topology; it is also linear continuum)
(promised this theorem as an application of compactness)

pf. assume $[0,1]^2$ were metrizable
define $f:[0,1]\to\mathbb{R}$ by $f(x)=\text{diam}(\{x\}\times[0,1])>0$
(w.l.o.g. bound the metric to avoid infinite)
diam $(A)=\sup\{d(x,y):A\}$
Consider $x_0\in[0,1]$. Let $x_n\searrow x_0$ Then $(x_n,1)\to(x_0,1)$
and in fact for any sequence $x'_n\in\underbrace{[(x_0,1),(x_n,1)]}_{D_n}$
$x'_n\to(x_0,1)$
Thus $\lim\limits_{n\to\infty}\overline{d}((x_0,1),D_n)=0$
$\overline{d}(x,A)=\sup\{d(x,y):y\in A\}$
diam $(D_n)\le 2\overline{d}((x_0,1),D_n)$ by triangle inequality
So $\text{diam}(D_n)\to 0$ and $D_n\supset\{x_n\}\times[0,1]$
$\Rightarrow f(x_n)=$ diam $(\{x_n\}\times[0,1])\to 0$

So $\lim\limits_{x\to x_0^+} f(x) = 0$

Similarly let $x_n \nearrow x_0$ argue with $(x_0, 0) \leftarrow (x_n, 0)$
that $\lim\limits_{x\to x_0^-} f(x) = 0$. Thus now
$\forall x_0 \in [0,1] \quad \lim\limits_{x\to x_0} f(x) = 0, \text{ but } f(x_0) > 0$

<u>Claim</u> $\nexists$ such $f \Rightarrow \blacksquare$

<u>*pf.*</u> Let $\epsilon > 0$
If exist infinitely many $x \in [0,1]$ with $f(x) \geq \epsilon$
then by compactness they have an accumulation point
$\exists x_n \to x_0 \in [0,1] \quad x_0 \neq x_n$
$f(x_n) \geq \epsilon \quad \lightning$ to $\lim\limits_{x\to x_0} f(x) = 0$
So $\forall \epsilon \quad S_\epsilon = |\{x : f(x) \geq \epsilon\}| < \infty$
Then $S = |\{x : f(x) > 0\}| = \bigcup\limits_{i=1}^{n} S_{1/n}$ is countable,
but we wanted $S = [0,1]$, which is uncountable $\lightning\blacksquare$

<u>HW</u> do it for $[0,1] \times [0,1)$; similar proof for $\{f : \mathbb{R} \to \mathbb{R}\}$ product topology
<u>but</u> $\{f : \mathbb{Z}_+ \to \mathbb{R}\} = \mathbb{R}^\omega$ with product topology is metrizable Th. 20.5 p.123

Covering Compactness (sec. 26)

<u>**Definition**</u> (X, d) metric space
$\mathcal{O} \subset \mathcal{P}(x)$ is an open cover(ing) of X if
$\bigcup \mathcal{O} = X$ and $\forall O \in \mathcal{O} \quad O$ open in X
$\mathcal{O}' \subset \mathcal{P}(X)$ is called a subcover of $\mathcal{O}$ if $\mathcal{O}' \subset \mathcal{O}$ and $\mathcal{O}'$ cover of X

<u>**Definition**</u> X is <u>covering compact</u> if every open cover of X contains a finite subcover

<u>**Theorem**</u> (the Borel-Lebesgue Covering Theorem)
(X, d) metric space X sequentially compact $\Leftrightarrow$ X covering is compact
for the proof, we need lemmas

<u>**Lemma**</u> (X, d) compact space $\Rightarrow \forall X_1 \supset X_2 \supset X_3 \supset \dots$ filtration with
nonempty <u>closed</u> sets, we have $\bigcap\limits_{i=1}^{\infty} X_i \neq \varnothing$

<u>*pf.*</u> Take $x_i \in X_i \quad \exists x_{n_i} \to x \quad X_i$ closed $\quad x \in X_i \ \forall i \quad \blacksquare$

<u>**Remark**</u> $\Leftarrow$ also true. See last part of B-L proof below

<u>**Ex.**</u> $X = \mathbb{R} \quad x_i = (-\infty, -i]$ closed filtration $\quad \bigcap X_i = \varnothing$

<u>**Lemma**</u> (HW) $x_n \to x$ in $(X, d) \quad \forall \epsilon > 0 \quad \exists N \quad \forall n, m \geq N$
$d(x_n, x_m) < \epsilon$ (Cauchy-property) $\blacksquare$

Pf of B-L "$\Rightarrow$" Prof. Friedrich 2.11.90

Step.1 X sequentially compact $\Rightarrow \exists$ finite ϵ-net
Step.2 $\Rightarrow X$ separable (HW)
Step.3 $\Rightarrow X$ is Lindelöf (every cover has <u>countable</u> subcover)
Step.4 $\Rightarrow X$ covering compact

<u>Step.1</u> X compact $\Rightarrow \forall \epsilon > 0 \quad \exists p_1, \dots, p_k \in X$

$$\bigcup_{i=1}^{k} B_\epsilon(p_i) = X \qquad =: \{p_1, \dots, p_k\} \text{ is } \underline{\epsilon\text{-net}}$$

<u>*pf.*</u> by $\lightning$ assume $\exists \epsilon > 0 \quad \forall p_1, \dots, p_k \quad \forall k \quad \{p_1, \dots, p_k\}$ not ϵ-net
take $p_0 \in X \quad B_\epsilon(p_0) \subsetneq X \quad \exists p_1 \notin B_\epsilon(p_0)$
$B_\epsilon(p_0) \cup B_\epsilon(p_1) \subsetneq X \quad \exists p_2 \notin B_\epsilon(p_0) \cup B_\epsilon(p_1)$
So find a sequence $p_i \subset X$ with $d(p_i, p_j) > \epsilon \quad \forall i, j$
any subsequence gives $\lightning$ to Cauchy-property $\Rightarrow$ does not converge.

<u>Step.2</u> X compact (sequentially) $\Rightarrow \exists A \subset X \quad |A| \le \omega \quad \overline{A} = X \quad X$ separable

$$A = \bigcup_{n=1}^{\infty} \{\text{finite } 1/n\text{-net}\}$$ (think about it!) used $d(x, A)$ lemma

<u>Step.3</u> X separable, $\mathcal{O}$ open cover
to show $\exists \mathcal{O} \supset \mathcal{O}'$ countable subcover
Let $A \subset X,\ |A| \le \omega,\ \overline{A} = X$
$\Lambda = \{(a, r) \in A \times \mathbb{Q} : \exists O \in \mathcal{O}\ B_r(a) \subset O\}$

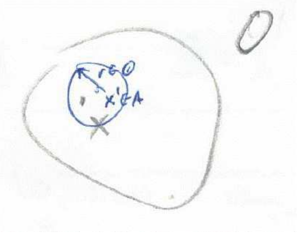


Figure 2 - 37

Λ countable
for each $(a, r) \in \Lambda$ choose an $O = O(a, r) \in \mathcal{O}$ with $B_r(a) \subset O$ (AC!)

$\mathcal{O}' = \{O(a,r) : (a,r) \in \Lambda\} \quad |\mathcal{O}'| \leq \omega$

<ins>claim</ins> $\mathcal{O}'$ is open cover of X (subcover of $\mathcal{O}$), i.e. $\bigcup \mathcal{O}' = X$

<ins>*pf.*</ins> $x \in X \quad \mathcal{O}$ cover $\exists O \in \mathcal{O} \quad x \in O \quad \exists \epsilon > 0 \quad B_\epsilon(x) \subset O$

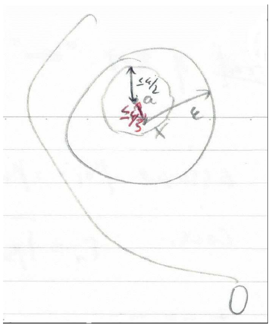


Figure 2 - 38

Since A is dense, $\exists a \in A \quad d(x,a) < \epsilon/3$

$\Rightarrow$ take $r \in \mathbb{Q} \cap (\epsilon/3, \epsilon/2)$

$r > \epsilon/3 \quad B_r(a) \ni x$

$r < \epsilon/2 \quad B_r(a) \subset B_{\epsilon/3+\epsilon/2}(x) \subset B_\epsilon(x) \subset \mathcal{O}$

$\exists O \in \mathcal{O} \quad B_r(a) \subset O$

$(a,r) \in \Lambda \quad x \in B_r(a) \subset O' := O(a,r) \in \mathcal{O}'$

$\forall x \in X \exists O' \in \mathcal{O}' : \quad x \in O' \quad \blacksquare$

<ins>Step.4</ins> (final) $\mathcal{O}$ open cover

$\exists \mathcal{O}'$ countable subcover $\quad X = O_1 \cup O_2 \cup \dots \cup \dots$

Consider $F_n = X \backslash \bigcup_{i=1}^{n} O_i$

$F_1 \supset F_2 \supset \dots \quad F_i$ closed

if $\forall i F_i \neq \varnothing \xRightarrow[\text{Lemma}]{} \bigcap_{i=1}^{\infty} F_i \neq \varnothing \Rightarrow \bigcup_{i=1}^{\infty} O_i \neq X ↯$

So some $F_n = \varnothing \Rightarrow X = \bigcup_{i=1}^{n} O_i$

$\mathcal{O}' = \{O_i\}_{i=1}^{n}$ is finite subcover of $\mathcal{O} \quad \blacksquare$ (end of "$\Rightarrow$")

<ins>**Pf of B-L**</ins> "$\Leftarrow$"X convering compact $\Rightarrow X$ sequentially compact.

assume $p_1, \dots, p_n \dots$ sequence in X

Consider $F_n = \overline{\{p_{n+1}, p_{n+2}, \dots\}} \quad U_n = X \backslash F_n$

$F_1 \supset F_2 \supset F_3$ closed $\quad U_1 \subset U_2 \subset U_3$ open

if $\bigcup_{n=1}^{\infty} U_n = X \xRightarrow[\text{covering property}]{} \bigcup_{n=1}^{N} U_n = X \Rightarrow F_N = \varnothing ↯$

Thus $\bigcup_{i=1}^{\infty} U_i \subsetneq X \Rightarrow \bigcap_{i=1}^{\infty} F_i \neq \varnothing$

$p \in \bigcap_{i=1}^{\infty} F_i \quad p \in F_1 = \overline{\{p_2, p_3, \dots\}} \quad \epsilon = 1 \quad \exists n_1 > 1 \quad d(p_{n_1}, p) < 1$

$p \in F_{n_1} = \overline{\{p_{n_1+1}, \dots\}} \quad \epsilon = 1/2 \quad \exists n_2 > n_1 \quad d(p_{n_2}, p) < 1/2$

$p \in F_{n_2} = \overline{\{p_{n_2+1}, \dots\}} \quad \epsilon = 1/3 \quad \exists n_3 > n_2 \quad d(p_{n_3}, p) < 1/3$

$\dots p_{n_i} \to p$ subsequence of $(p_n) \quad \blacksquare$

(Do Heine's Theorem; 27.6, p.174)

Now to general topological spaces

Covering Compactness Section. 26 Limit point Compactness Section. 28

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Covering Compactness} \arrow[rr, Rightarrow, shift right] \arrow[d, Rightarrow, shift right, "\text{X, p.34}(*1)"', "/" marking] & & \text{Accumulation point Compactness} \arrow[ll, Rightarrow, shift right, "\text{X, Th. 28.1 p.177}(S_\Omega)"', "/" marking] \arrow[dll, Rightarrow, shift left, "\text{X p. 34}(*1)", "/" marking] \\
\text{Sequence Compactness} \arrow[u, Rightarrow, shift right, "\text{X,}S_\Omega"', "/" marking] \arrow[urr, Rightarrow, shift left, "\text{works as above}"] & &
\end{tikzcd}$$

Lemma order topology is covering compact $\Rightarrow \exists$ smallest and largest element.

Rem The converse is true if LUBP; see Th 27.1 in book

Theorem order topology s.t. $\exists$ smallest, largest element and Least upper bound property, $\Rightarrow$ Seqence compact (Generalization of Bolzano-Weierstrass theorem)

Ex. $S_\Omega = \overline{S}_\Omega \backslash \{\Omega\}$ "smallest uncountable ordered set"
but no largest element $\Rightarrow$ not covering compact
but S_Ω is sequentially (and hence accumulation point) compact $\Rightarrow$ not metrizable!
$x_1, x_2, \cdots \subset S_\Omega \quad \{x_i\} \subset S_\Omega \quad |\{x_i\}| \leq \omega \Rightarrow \{x_i\}$ bounded
a_0 smallest element of S_Ω $\exists b \in S_\Omega \quad \{x_i\} \subset [a_0, b]$ sequentially compact.
Generalization of Bolzano-Weierstrass $\Rightarrow \exists$ converging subsequence in $[a_o, b]$
also converging in S_Ω
(order topology of interval = relative topology)

Remark This example shows also if $x \in \overline{A}$ (pg.13) $A = S_\Omega$
then not necessarily $x \in A_{\text{lim}}$ $\Leftarrow x \in \overline{S}_\Omega \quad x = \Omega$
like in pg. 23
$\overline{A} = \overline{S}_\Omega$ but $\Omega \notin (S_\Omega)_{\text{lim}}$
$\uparrow$ explains the notation $\overline{S}_\Omega$

(∗1)
Theorem Tychonoff's Theorem (Section 37, p. 167)
The product topology of (any number of) (covering) compact spaces is (covering) compact

Ex. $\Sigma = \{0,1\}^{\mathbb{Z}_+} = \{$sequence of 0,1$\}$($\{$0,1$\}$ discrete topology)
$X = \{0,1\}^{\Sigma}_{\text{prod}}$ is (c.) compact by Tychonoff's theorem
($\Rightarrow$ also accumulation point compact) (X is T_2 etc.)
but X is not sequentially compact.

Pf.

$$\begin{tikzcd} f_n \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & g \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ \{0,1\}^{\Sigma} & \{0,1\}^{\Sigma} \end{tikzcd} \qquad \forall \sigma \in \Sigma \quad \exists n \;\; f_i(\sigma) = g(\sigma) \;\; i \geq n$$

pointwise convergence

$\Rightarrow \{(f_n)$converge $\Leftrightarrow \forall \sigma \in \Sigma \;\; \{f_n(\sigma)\}$ is eventually constant$\}$

Consdier $(f_n) \subset \{0,1\}^{\Sigma}$ given by $f_n(\sigma) = \sigma(n)$

$f_n : \Sigma \to \{0,1\} \qquad f_n((0,1,0,\dots,0,1,\underset{n}{1},0,\dots)) = 1$

Let $(f_{n_k}) \subset (f_n)$; choose $\hat{\sigma} \in \Sigma$ with $\hat{\sigma}(n_i) = \left\{ \begin{array}{ll} 1 & i \text{ even} \\ 0 & i \text{ odd} \end{array} \right\}$

$f_{n_k}(\hat{\sigma}) = \hat{\sigma}(n_k) = \left\{ \begin{array}{ll} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{array} \right\}$ is not eventually constant

$f_{n_k} \nrightarrow g$

Remark There are simpler examples for ‘acc. point compact $\not\Rightarrow$ sequentially compact’ like $X_{\text{disc}} \times \{0,1\}_{\text{indisc}}$, $|X| = \infty$, but not even T_0

Above (*1) has an example of a space that is covering (and accumulation point) compact, but not sequentially compact, but this example uses Tychonoff’s theorem

Pb. Prove that I_0^2 ordered space is covering compact

sol. $\mathcal{O}$ open cover

Let $x_0 \in [0,1]$ $\quad I(x_0) = x_0 \times [0,1]$

$\underline{\mathcal{O} \text{ induces covering of } I(x_0)}$ $\quad \tilde{O}_{x_0} = \{O \cap I(x_0) : O \in \mathcal{O}\}$

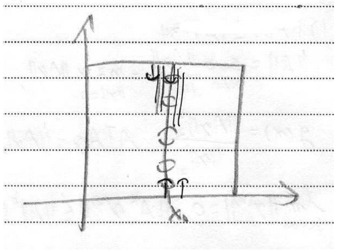


Figure 2 - 39

$I(x_0) \cong [0,1]$ (covering) compact

$\exists \tilde{\mathcal{O}}'_{x_0} \subset \tilde{O}_{x_0}$ $|\tilde{\mathcal{O}}'_{x_0}| < \infty$ $\quad \bigcup \tilde{\mathcal{O}}'_{x_0} = I(x_0)$

for each $\tilde{O}' \in \tilde{\mathcal{O}}'_{x_0}$ choose an $\hat{O}_{\tilde{O}'} \in \mathcal{O}$ with $\hat{O}_{\tilde{O}'} \cap I(x_0) = \tilde{O}'$

$\mathcal{O}_{x_0} = \{\hat{O}_{\tilde{O}'} : \tilde{O}' \in \tilde{\mathcal{O}}'_{x_0}\}$ $\quad |\mathcal{O}_{x_0}| < \infty$

$\Rightarrow \exists \mathcal{O}_{x_0} \subset \mathcal{O}$ $\quad |\mathcal{O}_{x_0}| < \infty$ $\quad (\Lambda_{x_0} :=) \bigcup \mathcal{O}_{x_0} \supset I(x_0)$

$\Lambda_{x_0} \ni x_0 \times 0$ open $\Rightarrow \Lambda_{x_0} \supset (x'_0 \times y'_0, x_0 \times 0]$ $\quad x'_0 < x_0$ $(x_0 \neq 0)$

$\Rightarrow \Lambda_{x_0} \supset (x'_0, x_0) \times [0,1]$

Similarly if $x_0 \neq 1$ $\quad \Lambda_{x_0} \ni x_0 \times 1$

$\Rightarrow \Lambda_{x_0} \supset (x_0, x''_0) \times [0,1]$ $\quad x''_0 > x_0$

$\Rightarrow \Lambda_{x_0} \supset (x'_0, x''_0) \times [0,1]$

so for each $x_0 \in [0,1]$ $\exists$ finite subfamily of $\mathcal{O}, \mathcal{O}_{x_0}$ and

$\exists x'_0 < x_0 < x''_0$ $\quad \bigcup \mathcal{O}_{x_0} \supset (x'_0, x''_0) \times [0,1]$

with $(x'_0, x''_0) = (x'_0, 1]$ $\quad x_0 = 1$

$= [0, x''_0)$ $\quad x_0 = 0$

Now $\{(x'_0, x''_0) \ : \ x_0 \in [0,1]\}$ is an open cover of $[0,1]$

$\xRightarrow{[0,1] \text{ compact}} \exists x_1, \ldots, x_n : (x'_1, x''_1) \cup \cdots \cup (x'_n, x''_n) = [0,1]$

then $\bigcup_{i=1}^{n} \mathcal{O}_{x_i}$ is a finite subcover of $\mathcal{O}$

Pb. Prove that I_0^2 is sequentially compact

sol. Let $(x_n \times y_n) \subset I_0^2$
$(x_n) \subset [0,1]$ sequence $\exists x_{n_k} \to x$
w.l.o.g. $(1) x_{n_k} \equiv x$ $(2)\ x_{n_k} \nearrow x$ $(3)\ x_{n_k} \searrow x$
go over to subsequence
$(2)\ (x_{n_k}, y_{n_k}) \to (x,0)$ $\qquad (3)\ (x_{n_k}, y_{n_k}) \to (x,1)$
$(1)\ \exists (y_{n_{k_l}}) \subset (y_{n_k})\ y_{n_{k_l}} \to y$ $\qquad (x_{n_{k_l}}, y_{n_{k_l}}) \to (x,y)$

The I_0^2 is sequentially compact
generalize that argument to prove the $[0,1]^\omega$ is sequentially compact with the dictionary order.
↳ (solution below)
(It is also covering compact but this needs Th 27.1 in book and HW 10.)

Sol. Assume $(f_k)_{k=1}^\infty \in \mathcal{F}(\mathbb{Z}_+, [0,1]) \simeq [0,1]^\omega$
does not have converging subsequence in dictionary order. We give $\lightning$.
We construct for each $n \in \mathbb{Z}_+$ a subsequence $(f_{k,n})_{k=1}^\infty$ of f_k
and a sequence $(C_k)_{k=1}^\infty$ such that $(f_{k,n+1}) \subset (f_{k,n})$ is a subsequence
and $f_{k,n}|_{\{1,\dots,n-1\}} = C|_{\{1,\dots,n-1\}}$.
We prove by induction, $i = 1$ is ok $f_{k,1} = f_k$
Assume $(f_{k,n})_{k=1}^\infty$ constructed as well as $C_1, \dots, C_{n-1}$

Consider $(f_{k,n}(n))_{k=1}^\infty$ w.l.o.g. $\exists (f_{k,n+1}) \subset (f_k)$

$$f_{k,n+1}(n) \to C_n \text{ w.l.o.g. either } \begin{array}{ll} (1) f_{k,n+1}(n) \equiv C_n & (k\to\infty) \\ (2) f_{k,n+1}(n) \nearrow C_n & (k\to\infty) \\ (3) f_{k,n+1}(n) \searrow C_n & (k\to\infty) \end{array}$$

$$(2)\ f_{k,n+1} \xrightarrow[k\to\infty]{} f(l) = \left\{ \begin{array}{cc} C_l & l \le n \\ 0 & l > n \end{array} \right\} \quad \lightning$$

$$(3)\ f_{k,n+1} \xrightarrow[k\to\infty]{} f(l) = \left\{ \begin{array}{cc} C_l & l \le n \\ 1 & l > n \end{array} \right\} \quad \lightning$$

(1) constructed $f_{k,n+1}$ and C_n Induction complete
So now $\exists (f_{k,n}) \subset (f_k)$ $(C_1, C_2, \dots)$ as needed.
Consider $(f_{k,k})_{k=1}^\infty$ diagonalization
$f_{k,k}(n) = C_n \quad k > n$
$f_{k,k} \xrightarrow[k\to\infty]{} (C_1, C_2, C_3, \dots) \simeq (f(n) = C_n)$ $\lightning$

Thus for non-metrizable spaces, '(any sort of) compact ⇒ separable ' is false!

23. Connected Spaces

Definition X topological space. A separation (I write (disjoint) decomposition)
$U, V \quad U, V \subset X$ open $\quad U, V \neq \varnothing$
$U \cup V = X \quad U \cap V = \varnothing$
X connected $\Leftrightarrow X$ has no separation
X disconnected $\Leftrightarrow \exists$ separation

Remark X is connected $\Leftrightarrow \forall A \subset X \quad A$ open and closed
$\Rightarrow A = \varnothing$ or $A = X$

Remark do not use "separable" in this context ($\exists$ separation) for disconnected $\exists A \subset X : \overline{A} = X, |A| \le \omega$ pg. 32 step 2. in B-L proof

Definition $A \subset X$ separation of A in X
$U, V \quad U, V \neq \varnothing \quad U, V$ not necessarily open
$A = U \cup V \qquad U \cap \overline{V} = \varnothing$
$\qquad\qquad\qquad \overline{U} \cap V = \varnothing$
U, V disjoint and none containing an accumulation point of the other

Lemma A is connected (in relative topology) $\Leftrightarrow$ A has no separation

Ex. $X = \mathbb{R} \quad A = [-1,0) \cup (0,1], (U = [-1,0), V = (0,1])$
$\overline{(0,1]} \cap [-1,0) = [0,1] \cap [-1,0) = \varnothing$
$(0,1] \cap \overline{[-1,0)} = (0,1] \cap [-1,0] = \varnothing$
A disconncected

Remark $\overline{U} \cap \overline{V} = \{0\} \neq \varnothing$ but this is not forbidden!
$\uparrow$ U, V have common accumulation point

Ex. $\mathbb{R}^2$

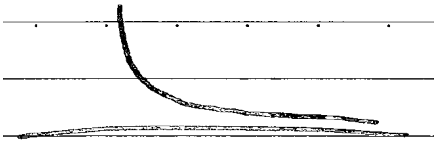


Figure 2 - 40

$\overset{U}{\{(x,0) : x \in \mathbb{R}\}} \cup \overset{V}{\{(x, 1/x) : x > 0\}}$
separation
$U \cap V = \overline{U} \cap \overline{V} = \varnothing$

Ex. $\mathbb{R}^2$

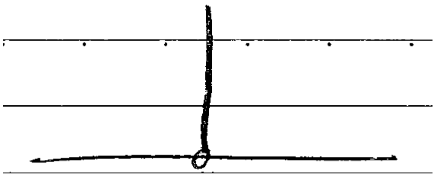


Figure 2 - 41

$\overset{U}{\{(x,0) : x \in \mathbb{R}\}} \cup \overset{V}{\{(0, 1/x) : x > 0\}}$
no separation
$\overline{V} \cap U \neq \varnothing$

Lemma C, D separation of $X \quad Y \subset X$ connected
$\Rightarrow Y \subset C$ or $Y \subset D$

(!)Theorem $X \ni x \quad \mathcal{Y} \subset \mathcal{P}(X) \quad \forall Y \in \mathcal{Y} \quad Y$ connected, $Y \ni x$
$\Rightarrow \bigcup \mathcal{Y}$ connected

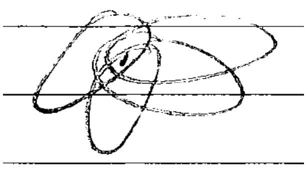


Figure 2 - 42

(*)Theorem If $A \subset X$ connected and $A \subset B \subset \overline{A} \Rightarrow B$ connected.

Ex. not true for interior!

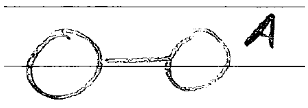


Figure 2 - 43

Theorem The image of a connected set under continuous map is connected

Theorem X, Y connected $\Rightarrow X \times Y$ connected
$\Rightarrow$ true for finite products

Ex. $\mathbb{R}^\omega$ box topology or uniform topology
$\mathbb{R}$ connected (prove later)
$\mathbb{R}^\omega = \{\text{bounded sequences}\} \cup \{\text{unbounded sequences}\}$
both open and disjoint non-empty
$\Rightarrow \mathbb{R}^\omega$ disconnected with uniform or box topology

Ex. $\mathbb{R}^\omega$ with product topology
$\mathbb{R}^n \simeq \{(x_1, \ldots, x_n, 0, \ldots, 0)\} \hookrightarrow \mathbb{R}^\omega$ connected
$\bigcap \mathbb{R}^n = 0 \quad R = \bigcup_{n=1}^{\infty} \mathbb{R}^n$ connected
$\mathbb{R}^\omega = \overline{R} \qquad f : \mathbb{Z}_+ \to \mathbb{R} \in \mathbb{R}^\omega$

$$R \text{ dense } f_n(x) = \left\{ \begin{array}{ll} f(x) & x \le n \\ 0 & x > n \end{array} \right\} \to f \text{ pointwise}$$

$\mathbb{R}^\omega$ connected

24. Connected Subspaces of $\mathbb{R}$

Definition (recall) $(X, <)$ ordered set is a linear continuum if
0) $|X| > 1$
1) Least Upper Bound Property(LUBP)
2) $\forall x, y \in X \quad x < y \Rightarrow \exists z \in X \quad x < z < y$
Intermediate Element Property(IEP)

(!)Theorem L is linear continuum with order topology, $Y \subset L$ convex
$\Rightarrow Y$ connected
in particular, L connected, and so are intervals and rays in L
recall: $Y \subset L$ convex $\forall x, y \in Y \quad x < y \quad \forall z \in L$
$x < z < y \Rightarrow z \in Y$

pf. By contradiction

Assume $Y = A \cup B \quad A \cap B = \varnothing \quad A, B \neq \varnothing \quad A, B$ open (in Y)

$a \in A \quad b \in B$ assume w.l.o.g. $a < b$

$I = [a,b] = \underbrace{A \cap [a,b]}_{A_0} \cup \underbrace{B \cap [a,b]}_{B_0} \quad A_0, B_0$ open in I

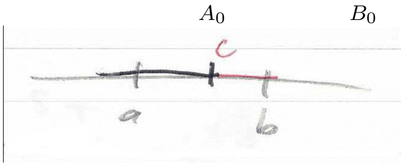


Figure 2 - 44

$c = \sup \underbrace{A \cap [a,b]}_{A_0}$

$\exists d < c$

1) If $c \in B_0 \Rightarrow c \neq a \Rightarrow (d,c] \subset B_0 \quad B_0$ open in I

$\exists d < f < c \quad [f,c] \subset B_0$

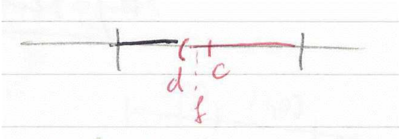


Figure 2 - 45

$[f,c] \cap A_0 = \varnothing$ $\lightning$ to $c = \sup A_0$

2) If $c \in A_0 \Rightarrow c \neq b \Rightarrow \exists [c,d) \subset A_0 \quad A_0$ open in I

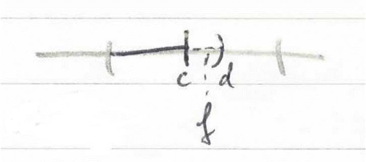


Figure 2 - 46

$\exists c < f < d \quad f \in A_0$ $\lightning$ to $c = \sup A_0$ ■

Corollary $\mathbb{R}$ connected and intervals and rays in it.

Remark Converse of this Theorem also true:
$(X, \mathcal{A}_<)$ connected $\Rightarrow (X,<)$ is linear continuum

Theorem Intermediate value theorem
$(X, \mathcal{A})$ connected $\quad (Y,<)$ order topology
$f : X \to Y$ continuous $\Rightarrow \forall x, y \in X \quad \forall c \in [f(x), f(y)]$
$\exists d \in X \quad f(d) = c$

pf. by contradiction if $\exists r \in (f(x), f(y)) \quad r \notin f(X)$
Consider $X = f^{-1}((-\infty, r)) \cup f^{-1}((r, \infty))$
separation of X

Remark take $X' = [x,y] \to d \in [x,y]$ "Darboux property"

Ex. I_0^2 ordered square with dictionary order is a linear continuum
different from $\mathbb{R}$

Ex. X well ordered $\Rightarrow X \times [0,1)$ with dictionary order is a linear continuum

Important case:

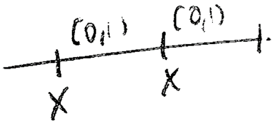


Figure 2 - 47

$X = S_\Omega (= \overline{S_\Omega} \setminus \{\Omega\})$
consider $S_\Omega \times [0,1) \setminus \{a_0 \times 0\} =: \Lambda$
with dictionary order
topologist's long line
Λ is locally homeomorphic to $\mathbb{R}$ (every point has a neighborhood homeomorphic to an interval)
but not embeddable in $\mathbb{R}$; it is not separable

25. Components

Definition X topological space define equivalence relation on X by $x, y \in X$
$x \sim y \quad \exists$ connected subspace W of $X \quad W \ni x, y$

equivalence class
Definition $[x]_\sim$ is the connected component of $x \in X$
$C_x = [x]_\sim = \bigcup\{C \subset X$ connected, $C \ni x\}$ connected by pg.38 (!)
maximal connected set containing x

Remark C_x closed as $\overline{C_x}(\ni x)$ is connected $\Rightarrow \overline{C_x} \subset C_x$

Definition X satisfies the connected neighborhood condition (CNC) (resp. at $x \in X$) if every point (resp. the point $x \in X$) has a connected neighborhood

Lemma $y \in C_x \quad \Rightarrow \forall (U,V)$ separation of $X \quad x \in U \Rightarrow y \in U$
$\Leftarrow$ if CNC see Ex. 3.26.10

pf. $\Rightarrow$ $\exists$ connected subspace $W \ni x, y$
assume by contradiction $\exists (U,V) \quad x \in U \quad y \in V$
$(U \cap W, V \cap W)$ separation of W
W connected $\Rightarrow U \cap W (\ni x) = \varnothing$ or $V \cap W (\ni y) = \varnothing$ $\lightning$

$\Leftarrow$ by contraposition $\exists x \not\sim y$
$\forall W \ni x$ connected $\quad W \not\ni y$
$C_x = \bigcup\{W \ni x \text{ connected}\} \not\ni y$
connected neighborhood condition $\Rightarrow C_x$ open(and closed)
(1) (2) or Remark above
$\Rightarrow \underset{\ni x}{C_x} \cup \underset{\ni y}{(X \setminus C_x)}$ is a separation

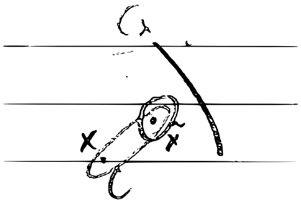


Figure 2 - 48: (1)

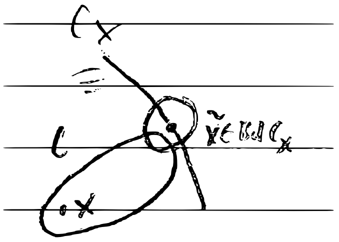


Figure 2 - 49: (2)

Note If $A \subset X$, connected components of A are meant with respect to relative topology

Definition X is totally disconnected if all its connected components are points

Example If $|S| = n < \infty$
$\mathcal{F}(S, \mathbb{R}_{\text{discr}})_{\text{prod}} \simeq \mathbb{R}^n_{\text{discr}}$ totally disconnected
$\mathcal{F}(\mathbb{Z}_+, \mathbb{R}_{\text{discr}})_{\text{prod}} = (\mathbb{R}_{\text{discr}})^\omega \overset{\text{why?}}{=} (\mathbb{R}^\omega,$ dictionary order) also totally disconnected

Remark $A \subset X$ discrete $\Rightarrow A$ totally disconnected (in relative topology)
converse is false

Ex. $\mathbb{Q} \subset \mathbb{R}$ is totally disconnected (but surely not discrete)
$x < y \quad \exists r \in (x, y)$ irrational
$\mathbb{Q} = \underbrace{(\mathbb{Q} \cap (-\infty, r))}_{\text{open in } \mathbb{Q}} \cup \underbrace{(\mathbb{Q} \cup (r, \infty))}_{\text{open in } \mathbb{Q}}$
I can separate points by open sets
(whose union is the whole space unlike in T_2!)

Remark $\mathbb{Q}$ does not satisfy the CNC, but "$\Leftarrow$" of lemma is still true

Remark CNC$\xRightarrow{(1),(2)}$ every union of connected components is open and closed
(which is false for $\mathbb{Q}$)

Definition X locally connected at x
$\forall U \ni x$ neighborhood $\exists U' \subset U \quad U' \ni x$ neighborhood connected

X locally connected if locally connected at x $\forall x \in X$

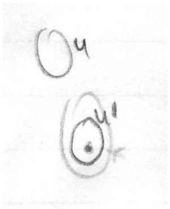


Figure 2 - 50

Remark X locally connected (at x)$\Rightarrow$ CNC (at x) (set $U = X$)

Theorem X is locally connected $\Leftrightarrow \forall U \subset X$ open
$\forall C_x \subset U$ component of U (in relative topology)
$C_x \subset X$ open

Ex. $\{f : \mathbb{R} \to \mathbb{R}\}$ with uniform topology $\mathcal{F}(\mathbb{R}, \mathbb{R})_{\text{uni}}$
$f(x) = x \in \{f : \lim_{x\to\infty} \frac{f(x)}{x} = 1\} = \Lambda_1$ open and closed
$0 \in \{f : \lim_{x\to\infty} \frac{f(x)}{x} = 0\} = \Lambda_0$
disconnected! every behavior → ∞ gives a separation

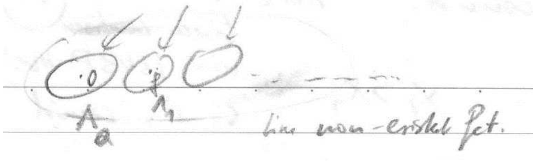


Figure 2 - 51

$\underbrace{\qquad\qquad\qquad\qquad}_{\{f:\mathbb{R}\to\mathbb{R}\}}$

connected components $[f]_\sim$

$f \sim g \Leftrightarrow f - g$ bounded

Path Connectedness and Path Components

Definition path: $[a,b] \to X$ from $x \in X$ to $y \in X$
is continuous $f(a) = x, f(b) = y$

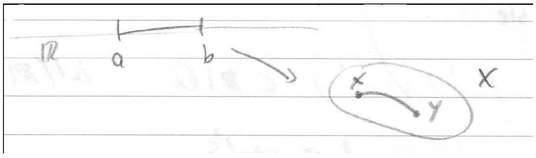


Figure 2 - 52

X is path connected if
$\forall x, y \in X \quad \exists$ path in X from x to y
write "$x \leftrightarrow y$"

Remark path-connected $\Rightarrow$ connected

Ex. $V, \|\cdot\|$ norm, whose unit ball $B = \{x \in V : \|x\| \le 1\}$
B is convex $\quad \forall x, y \in B,\ t \in [0,1], \quad tx + (1-t)y \in B$. Then
$t \mapsto tx + (1-t)y$ continuous in t (in norm topology)
B is path connected

Ex. $\mathbb{R}^n \backslash \{0\}$ path connected

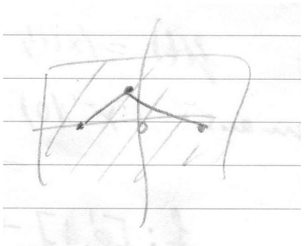
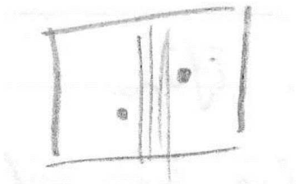


Figure 2 - 53

Ex. I_0^2 is not path connected, but it's a linear continuum (LUBP HW!) So it's connected



$a = (x_1, y_1) \quad b = (x_2, y_2) \quad x_2 > x_1$

Figure 2 - 54

$f : [0,1] \to X$ continuous image must be connected
if image $\ni a, b \Rightarrow$ image $\supset [a,b] \supset \underbrace{(x_1, x_2)}_{\text{uncountable}} \times [0,1]$

$\exists x_0 \in (x_1, x_2) \quad f(\underbrace{\mathbb{Q} \cap [0,1]}_{\text{countable}}) \cap (\{x_0\} \times \underset{\text{open}}{(0,1)}) = \varnothing$

$f^{-1}(\{x_0\} \times (0,1)) \subset \mathbb{R} \setminus \mathbb{Q}$, but $\quad \mathrm{Int}(\mathbb{R} \setminus \mathbb{Q}) = \varnothing$ ↯
non-empty open as f is continuous

Ex. $\underline{A = \{x \times \sin\frac{1}{x} : 0 < x < \frac{1}{\pi}\} \subset \mathbb{R}^2}$ topologist's sine curve

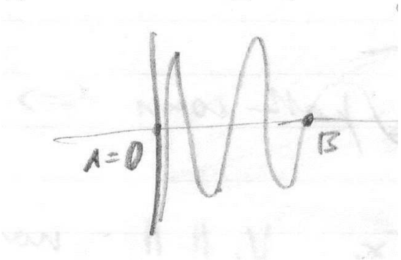


Figure 2 - 55

$\overline{A} = A \cup (\{0\} \times [-1,1])$
A connected $\Rightarrow \overline{A}$ connected but $\overline{A}$ is not path connected
Let there be a path $f : [0,1] \to \overline{A} \subset \mathbb{R}^2$
$f(t) = (x(t), y(t))$ with $f(0) = \underbrace{(0,0)}_{=A}$ $f(1) = \underbrace{(\frac{1}{\pi}, 0)}_{=B}$
x continuous $x^{-1}(0) \subset [0,1]$ closed $\Rightarrow \exists \max x^{-1}(0) =: a'$
w.l.o.g. $f : [a',1] \to \overline{A}$ $x(t) > 0$ $t > a' \Rightarrow$ $\text{img}(f|_{(a',1]}) \subset A$
$x(a') = 0, x(1) = \frac{1}{\pi} \Rightarrow$ by Intermediate Value Theorem
$\forall x \in (0, \frac{1}{\pi}]$ $\exists t$ $x(t) = x \Rightarrow y(t) = \sin(\frac{1}{x})$ $\text{Image}(f) = A \cup \{0 \times y(a')\}$
$[a',1]$ (sequentially) compact, f continuous $\Rightarrow \text{Img}(f)$ closed
$\Rightarrow$ $\text{Img}(f) = \overline{A \cup \{0 \times y(a')\}} = \overline{A} \neq A \cup \{0 \times y(a')\}$ $\lightning$
This means the closure of a path-connected set is not path-connected!
(Compare theorem(*) on pg. 38)

Definition X topological space, define equivalence relation by
$x \sim y$ $\exists$ path in X from x to y
$[x]_\sim$ is x's path component

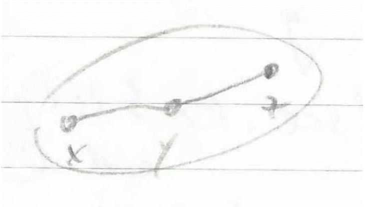


Figure 2 - 56

Theorem path components of X are disjoint path connected subspaces whose union is X
each path-connected subspace of X is contained in exactly one path component

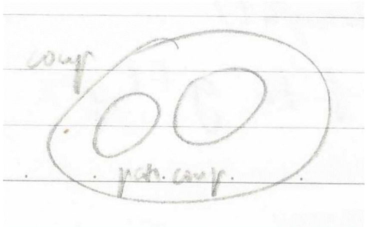
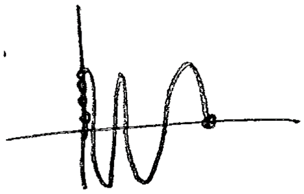


Figure 2 - 57

Ex. $\mathbb{Q} \subset \mathbb{R}$
all components trivial $\Rightarrow$ all path components are trivial

Ex. from pg.44 topologist's sine curve is connected but not path-connected!
A has one component, $\overline{A}$, and
two path components $\underset{=A_1}{A\backslash\{0\times 0\}}$ and $\underset{=A_2}{\{0\}\times[-1,1]}$
note that in A,
A_1 is open but not closed ⇒ unlike components, path
A_2 is closed but not open   components need not be closed

Ex. connected take $A\backslash\{0\times 0\}\cup\{0\}\times([-1,1]\backslash\mathbb{Q})$



this is still connected (= one component)
but has uncountably many path components!

Figure 2 - 58

Ex. Consider $\{f:\mathbb{R}\to\mathbb{R}\}$ with uniform topology
connected component C_f of f is the set of maps g
with $|g-f|$ bounded. (see Ex. 3.26.2 p.160 in book, p. 38 in note)
This is path connected in fact it is convex (as $\subset$ VS/$\mathbb{R}$)
$tf+(1-t)g$ gives a (straight) path from f to $g\in C_f$
Similarly you can do in box topology with
$C_f=\{g:|\{x:g(x)\neq f(x)\}|<\infty\}$ (is also convex)

Again there is a local version

Definition X is locally path connected if
$\forall U\ni x$ open $\exists U\supset U'\ni x\quad U'$ path connected

Remark locally path connected ⇒ locally connected

Ex. $(0,1)\times[0,1)$ with dictionary order is not connected, not locally connected
(⇒ not path connected and not locally path connected.)

Ex. $[0,1]\times(0,1)$ with dictionary order is not connected but locally connected
not path connected, but locally path connected

Ex. every set with the discrete topology is locally connected
and locally path connected since $U=\{x\}\ni x$ is open
(but, of course, totally disconnected and path-disconnected)

Theorem X locally (path) connected $\Leftrightarrow\forall U\subset X$ open
$\forall U'$ (path) component of $U\quad U'\subset X$ open

Theorem each path component of X lies (entirely) within a component of X
if X is locally path connected, then components and path components are the same

Corollary connected and locally path connected ⇒ path connected

Ex. $I_0^2 = [0,1]^2$ has LUBP(HW1) (See Pb.3. p.160 in book)
Intermediate Element Property (IEP) $\Rightarrow$ Linear continuum
$\xRightarrow{\text{Th.(!) p.39}}$ connected and locally connected

path connected?

$(a,b) \leftrightarrow (c,d)$ if $a = c$

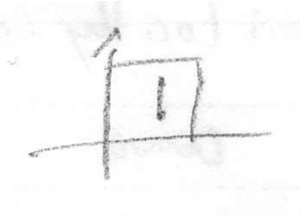


Figure 2 - 59

but $(a,b) \not\leftrightarrow (c,d)$ if $a \neq c$
assume $f : [0,1] \to I_0^2$ $\quad f(0) = (a,b) = a \times b$
$f(1) = (c,d) = c \times d$
$f([0,1]) \supset [a \times b, c \times d]$
$A = \mathbb{Q} \cap [0,1] \subset \overline{[0,1]}$ dense, countable
f continuous $\Rightarrow \overline{f(A)} \supset f(\overline{A}) = f([0,1]) \supset [a \times b, c \times d]$
$B := f(A) \cap (a \times b, c \times d)$ dense in $(a \times b, c \times d)$, countable
but $(a \times b, c \times d) \supset \bigcup\limits_{a<x<b} \{x\} \times [0,1] \Rightarrow \exists s \in (a,b)$
uncountable, disjoint union
$B \cap (\{s\} \times [0,1]) = \varnothing \quad (a \times b, c \times d) \supset \{s\} \times [0,1] \supset \{s\} \times (0,1)$ open
$\overline{B} \subsetneq (a \times b, c \times d)$ ↯

so path components of I_0^2 are $\{s\} \times [0,1]$

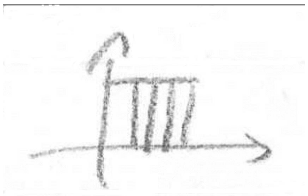


Figure 2 - 60

43. Complete metric spaces

Assume (X,d) metric space
Recall (proof of Borel-Lebesgue) $\quad (X,d) \supset (x_n)$

$$x_n \to x \Rightarrow \forall \epsilon\ \exists N\ \forall m,n \geq N\ d(x_m,x_n) < \epsilon \tag{$\cdot$}$$

Definition We say (x_n) is Cauchy-sequence if (x_n) satisfies $(\cdot)$

Lemma (x_n) converges $\Rightarrow (x_n)$ Cauchy-sequence

Definition We say (X,d) is complete if every Cauchy sequence converges
(i.e. converse of lemma holds)

Lemma Let (x_n) be a Cauchy-sequnece, if (x_n) has a convergent subsequence, then (x_n) converges

Corollary X complete $\Leftrightarrow$ every Cauchy sequence has a convergent subsequence

(‡)Comment $\underset{\substack{\text{Eucl.}\\ \text{non-complete}}}{(0,1)} \simeq \underset{\substack{\text{Eucl.}\\ \text{complete}}}{\mathbb{R}}$ homeomorphic

so $\exists$ metric on (0,1) giving Euclidean topology
with respect to which (0,1) is complete
Thus completeness depends on metric (not only on metrizability)

Corollary

X compact	$\Rightarrow$	X complete
topological property		metric property
metrizable topology	$\Rightarrow$	complete with respect to every metric
space compact		inducing the topology

Theorem $\mathbb{R}^k$ complete. (with Euclidean metric)

pf. let (x_n) be Cauchy-sequence
$\Rightarrow (x_n)$ bounded $\Rightarrow (x_n) \supset [-M, M]^k$ compact
$\Rightarrow (x_n)$ has convergent subsequence $\Rightarrow (x_n)$ converges

For $\mathbb{R}^\omega$ recall the following lemma

Lemma let $X = \prod_{\alpha \in I} X_\alpha$ with product topology, and
for $\alpha_0 \in I$ let $\pi_{\alpha_0} : X \to X_{\alpha_0}$
be the projection $(x_\alpha)_{\alpha \in I} \mapsto x_{\alpha_0}$
Then $\overline{x_n} \to \overline{x}$ in $X \Leftrightarrow \forall \alpha \in I\ \pi_\alpha(\overline{x_n}) \to \pi_\alpha(\overline{x})$ in X_α
(i.e. convergence in the product topology is pointwise convergence)

Theorem The product topology on $\mathbb{R}^\omega$ has a metric with respect to which it is complete

pf. $d(\overline{x}, \overline{y}) = \sup_{i>0} \left\{ \frac{\overline{d}(x_i, y_i)}{i} \right\}$ $\quad \overline{d}(a, b) = \min(|a - b|, 1)$
d gives product space
assume $(\overline{x_n})$ Cauchy-sequence in $(X = \mathbb{R}^\omega, d)$
$\forall i \quad \pi_i(\overline{x_n})$ is $\mathbb{R}$ is Cauchy-sequence because
$|\pi_i(\overline{a}) - \pi_i(\overline{b})| \leq i \cdot d(\overline{a}, \overline{b})$ $\quad i$-fixed
so $\pi_i(\overline{x_n}) \to a_i$ convergent in $\mathbb{R} \forall i$ Then,
$\overline{x_n} \to (a_i)_{i=1}^\infty$ in X

Remark (x_n) Cauchy-sequence, with respect to d
$\Leftrightarrow (x_n)$ Cauchy sequence with respect to $\overline{d}$
(X, d) complete $\Leftrightarrow (X, \overline{d})$ complete

Ex. $\mathbb{Q}, (-1, 1)$ with Euclidean metric not complete
consider $x_n \in \mathbb{Q} \quad x_n \to \sqrt{2}$ in $\mathbb{R}$ (for $\mathbb{Q}$)
$-1 + \frac{1}{n} \in (-1, 1) \to -1 \quad$ (for $(-1, 1)$)

Remark $\underset{\text{not complete}}{(-1,1)} \simeq \underset{\text{complete}}{\mathbb{R}}$ homeomorphic
so $\exists \overline{d}$ on (-1,1) giving Euclidean topology
with respect to which is complete → completeness
pg.47 (‡)Comment ⇐ depends on metric(not only on metrizability)

Theorem Let (X,d) complete.
$A \subset X$ closed $\Rightarrow (A, d|_{A\times A})$ complete

pf. $(x_n) \subset A \quad (x_n) \subset X$ Cauchy-sequence $(x_n) \to x$ in X
Cauchy-sequence A closed $\Rightarrow x \in A \Rightarrow$
$(x_n) \to x$ in A

Remark ⇐ also true: $A \subset X$ complete $\Rightarrow A$ closed

Remark $\mathbb{R}^J = \mathcal{F}(J,\mathbb{R})_{\text{prod}}$ is in general not metrizable,
so completeness makes no sense
but $\mathbb{R}^J_{\text{uni}}$ is metrizable. Recall <u>uniform metric</u>

Definition Let (Y_α, d_α) metric space.
Let $\overline{d_\alpha} = \min(d_\alpha, 1)$
For $Y = \prod_{\alpha\in J} A_\alpha$ define the uniform metric on Y by
$\overline{\varrho}(\overline{x},\overline{y}) := \sup_{\alpha\in J}\{\overline{d_\alpha}(\pi_\alpha(\overline{x}),\pi_\alpha(\overline{y}))\} \qquad \pi_\alpha : Y \to A_\alpha$

Theorem (Y,d) compete $\Rightarrow Y^J = \prod_{\alpha\in J} Y_\alpha \qquad Y_\alpha = Y$
is complete with uniform metric ↑ index the copy

pf. let $(f_n) \subset Y^J$ Cauchy-sequence with respect to $\overline{\varrho}$
then $(\pi_\alpha(f_n)) \subset Y_\alpha$ Cauchy-sequence in $(Y_\alpha, \overline{d}_\alpha)$
$\Rightarrow$ Cauchy sequence in (Y_α, d_α)
$\pi_\alpha(\overline{f_n}) \to y_\alpha$ in Y_α
Let $f : \alpha \mapsto y_\alpha$
We claim $f_n \to f$ in $(Y^J, \overline{\varrho})$
Given $\epsilon > 0$ choose N with $\forall n,m \geq N$
$\overline{d}(f_n(\alpha), f_m(\alpha)) < \epsilon/2 \quad \forall \alpha \in J (\Leftarrow \overline{\varrho}(f_m, f_n) < \epsilon/2)$
$\xrightarrow[\overline{d}\text{ continuous}]{m\to\infty} \overline{d}(f_n(\alpha), f(\alpha)) \leq \epsilon/2$
This holds $\forall \alpha \in J \quad \forall n \geq N$
$\overline{\varrho}(f_n, f) = \sup_\alpha \overline{d}(f_n(\alpha), f(\alpha)) \leq \epsilon/2 < \epsilon$
$\Rightarrow \forall \epsilon > 0\ \exists N\ \forall n \geq N\ \overline{\varrho}(f_n, f) < \epsilon \Rightarrow f_n \to f$

Definition Now assume X is topological space
$C(X,Y) \subset \mathcal{F}(X,Y) = Y^X$
$\shortparallel$
is $\{f \in Y^X : f \text{ continuous}\}$

Definition $f : X \to (Y,d)$ bounded if $f(X) \subset Y$
is a bounded set $\operatorname{diam}(f(X)) < \infty$
$B(X,Y) \subset Y^X$
$\shortparallel$
$\{f : X \to Y\ f \text{ bounded}\}$

Theorem (Y,d) metric space
$B(X,Y)$ and $C(X,Y)$ (X topological space)
are closed in $(Y^X, \overline{\varrho})$ and therefore complete

Ex. $C(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})_{\text{uni}}$ $\underset{\Downarrow}{\text{closed}}$ (but not discrete)
$C(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})_{\text{box}}$ (closed and) discrete $\Leftarrow$ HW
$C(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})_{\text{prod}}$ is dense (and not closed)
$\overset{\cup}{\mathbb{R}[z]}$ dense (Lagrange Interpolation)

Completion

Definition $(X,d), (Y,\tilde{d})$ metric spaces
we say $f: X \to Y$ is isometry if
$\forall x, \tilde{x} \in X \quad d(x,\tilde{x}) = \tilde{d}(f(x), f(\tilde{x}))$

Remark f isometry $\Rightarrow$ injective
so f is also called an "isometric embedding"

Theorem (Existence of completion)
(X,d) metric space $\exists (Y,\tilde{d})$ complete metric space
$f: X \to Y$ isometric embedding

Definition If (X,d) metric space $(Y,\tilde{d})$ compete metric space
$f: X \to Y$ isometric embedding
call $\overline{f(X)} \subset Y$ the completion of X

Remark completion is unique up to isometry

Construction $U(X^\omega) = \{(x_1, x_2, x_3, \dots) \in X \text{ Cauchy sequence}\} \subset X^\omega$
Let $\sim$ be equivalence relation on $U(X^\omega)$
$(x_i) \sim (x_i') :\Leftrightarrow d(x_i, x_i') \to 0$
Then $Y = \Gamma(X) := U(X^\omega)/\sim$
$\tilde{d}([(x_i)], [(x_i')]) = \lim_{i \to \infty} d(x_i, x_i')$
$f: X \hookrightarrow Y$ is given by $x \mapsto [(x, x, x, \dots)]$

Ex. $X = \mathbb{Q}$ with Euclidean metric $\Gamma(\mathbb{Q}) = \mathbb{R}$ construction of real numbers