GS2004 Linear Algebra Fall 2018

Jordan normal form and Jordan basis

Let $\mathbf{F}$ be a field. A *Jordan box* $J_{\lambda,n}$ (for $\lambda \in \mathbf{F}$ and $n \in \mathbb{N}_+$) is an $n \times n$ matrix over $\mathbf{F}$ of the sort

$$J_{\lambda,n} = \begin{bmatrix} \lambda & 1 & & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

We say that $A' \in M_n(\mathbf{F})$ is in *Jordan normal form* if it looks like

$$A' = \left[\begin{array}{c|c|c|c} J_{\lambda_1,n_1} & 0 & 0 & 0 \\ \hline 0 & J_{\lambda_2,n_2} & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & J_{\lambda_k,n_k} \end{array}\right]. \tag{1}$$

It is not necessary that λ_i are all different.

We say that $A \in M_n(\mathbf{F})$ *has a Jordan normal form*, if there is an ordered basis β, called *Jordan basis* for A, so that (for an invertible matrix Q),

$$A' := [L_A]_\beta = Q^{-1}AQ \tag{2}$$

is in Jordan normal form. Note that the columns of Q are the vectors of the Jordan basis.

It is very easy to see that then

$$\chi_A(t) = \chi_{A'}(t) = \prod_{i=1}^{k}(\lambda_i - t)^{n_i},$$

so that χ_A splits. The important is the converse of this easy fact.

Theorem 1. *Let $A \in M_n(\mathbf{F})$ be so that $\chi_A(t)$ splits. Then A has a Jordan normal form. This Jordan normal form is unique up to permuting the Jordan boxes J_{λ_i,n_i}.*

The proof of this theorem (in the book) is long and technical. In particular, it explains that every $A \in M_n(\mathbf{F})$ has a Jordan normal form when $\mathbf{F}$ is algebraically closed.

The case of non-splitting χ_A (over non-algebraically closed $\mathbf{F}$) is more complicated. For $\mathbf{F} = \mathbb{R}$, there is a version of the theorem as follows.

A *generalized Jordan box* $J_{a,b,n}$ for $a, b \in \mathbb{R}$, $b > 0$ (and $n \in \mathbb{N}_+$) is a $(2n) \times (2n)$ matrix over $\mathbb{R}$ of the sort

$$J_{a,b,n} = \left[\begin{array}{cc|cc|cc|cc|cc}
a & b & 1 & 0 & 0 & 0 & \cdots & & 0 & 0 \\
-b & a & 0 & 1 & 0 & 0 & \cdots & & 0 & 0 \\ \hline
0 & 0 & a & b & 1 & 0 & \cdots & & \vdots & \vdots \\
0 & 0 & -b & a & 0 & 1 & \cdots & & \vdots & \vdots \\ \hline
\vdots & \vdots & 0 & 0 & \ddots & \ddots & & & \vdots & \vdots \\
\vdots & \vdots & 0 & & & \ddots & \ddots & & \vdots & \vdots \\ \hline
0 & 0 & \cdots & & & 0 & a & b & 1 & 0 \\
0 & 0 & \cdots & & & 0 & -b & a & 0 & 1 \\ \hline
0 & 0 & & \cdots & & & 0 & 0 & a & b \\
0 & 0 & & \cdots & & & 0 & 0 & -b & a
\end{array}\right].$$

Note that $\chi_{J_{a,b,n}}(t) = ((t-a)^2 + b^2)^n$, i.e., $J_{a,b,n}$ corresponds to conjugate-complex roots $a \pm b\sqrt{-1}$ of χ_A.

We say that $A' \in M_n(\mathbb{R})$ is in *generalized Jordan normal form* if it looks like (1), except that beside J_{λ_k,n_k} we have diagonal boxes J_{a_l,b_l,n'_l}.

Theorem 2. *Let $A \in M_n(\mathbb{R})$. Then A has a generalized Jordan normal form $A' = [L_A]_\beta$ (again unique up to permuting, possibly generalized, Jordan boxes).*

Here we assume that $\mathbf{F} = \mathbb{R}$ but that χ_A splits, and explain how to find a(n ordinary) Jordan basis in practice. Note that, while the Jordan normal form is more-or-less unique, a Jordan basis is *very* non-unique! Thus we cannot expect to arrive at the same, or even very similar, results.

To understand how to find *one* Jordan basis, we first observe some of its properties. Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be such and

$$m_i = \sum_{j=1}^{i} n_j\,,$$

with $m_0 = 0$. Let for $j = 1, \dots, n_i$, $\mathbf{v}_{i,j} = \mathbf{v}_{m_{i-1}+j}$. Let $\beta_i = \{\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,n_i}\}$ (as on ordered set), and $V_i = \operatorname{span} \beta_i$. Then V_i are L_A invariant subspaces, and

$$\left[L_A \Big|_{V_i} \right]_{\beta_i} = J_{\lambda_i, n_i}\,.$$

Then one sees that

$$(L_A - \lambda_i Id)(\mathbf{v}_{i,j}) = \begin{cases} \mathbf{v}_{i,j-1} & j > 1 \\ 0 & j = 1 \end{cases}$$

The sequence $\mathbf{v}_{i,n_i} \to \mathbf{v}_{i,n_i-1} \to \cdots \to \mathbf{v}_{i,1} \to 0$ (where $\to$ is the application of $L_A - \lambda_i Id$) is called a *chain*. Thus a Jordan basis consists of chains. They form bases of the *generalized eigenspaces* $\operatorname{Ker}((L_A - \lambda_i Id)^j)$. Each chain contains exactly one eigenvector, thus the dimension of the eigenspace to eigenvalue λ is equal to the number of chains to eigenvalue λ.

Finding the chains must be done for each eigenvalue separately. The most complicated situation is when there are several chains (and Jordan boxes) to the same eigenvalue, and in particular when there are such of different length.

We will do this in the following example.

Problem 1. Find the Jordan normal form and a Jordan basis of the matrix

$$A = \begin{bmatrix} 157 & -91 & -11 & 3 \\ 192 & -112 & -13 & 5 \\ 597 & -341 & -45 & 2 \\ -150 & 83 & 13 & 4 \end{bmatrix}.$$

Hint: χ_A splits and all eigenvalues are integers.

Solving this requires some understanding, so be careful about what is being done. Essentially all we learned about linear algebra calculations is needed!

Solution. First, we determine the characteristic polynomial. (You must know how to do it, so I don't explain.) We find

$$\chi(t) = t^4 - 4t^3 + 6t^2 - 4t + 1\,.$$

Next, we find the eigenvalues, which are roots of χ. In general this can be difficult, but with the hint that χ_A splits all λ are integers, we need to try only divisors of the absolute term, which are ± 1. This way we find the factorization

$$\chi(t) = (t-1)^4\,,$$

which means there is only one eigenvalue, $\lambda = 1$.

Alternatively, we need not calculate χ_A, but only $\det A$, but then we need to test for $\lambda \mid \det A$ if $A - \lambda Id$ is invertible.

Next, we find the eigenspace, which is the kernel of $A - \lambda Id = A - Id$. We do row reduction. Let us fix the following notation.

- An entry '$\leftrightarrow [m]$' right to a row of a matrix means that in the following matrix, the row is exchanged with row m (type I operation)
- An entry '$\cdot \mu$' or '$: \mu$' right to a row of a matrix means that in the following matrix, this row is multiplied, resp. divided by μ (type II operation).
- An entry '$+\mu[m]$' right to a row of a matrix means that in the following matrix, μ times row m of the current matrix is added (type III operation)

$$\begin{bmatrix} 156 & -91 & -11 & 3 \\ 192 & -113 & -13 & 5 \\ 597 & -341 & -46 & 2 \\ -150 & 83 & 13 & 3 \end{bmatrix} \begin{matrix} +[4] \\ \\ -3[2] \\ \\ \end{matrix} \qquad \text{(try first to get the numbers smaller)}$$

$$\begin{bmatrix} 6 & -8 & 2 & 6 \\ 192 & -113 & -13 & 5 \\ 21 & -2 & -7 & -13 \\ -150 & 83 & 13 & 3 \end{bmatrix} \begin{matrix} :2 \\ +[4] \\ \\ \\ \end{matrix}$$

$$\begin{bmatrix} 3 & -4 & 1 & 3 \\ 42 & -30 & 0 & 8 \\ 21 & -2 & -7 & -13 \\ -150 & 83 & 13 & 3 \end{bmatrix} \begin{array}{r} \\ -14[1] \\ -7[1] \\ +50[1] \end{array} \tag{3}$$

$$\begin{bmatrix} 3 & -4 & 1 & 3 \\ 0 & 26 & -14 & -34 \\ 0 & 26 & -14 & -34 \\ 0 & -117 & 63 & 153 \end{bmatrix} \begin{array}{l} \\ :2 \\ -[2] \\ +9/2[2] \end{array}$$

$$\begin{bmatrix} 3 & -4 & 1 & 3 \\ 0 & 13 & -7 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \cdot 13 \\ \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 39 & -52 & 13 & 39 \\ 0 & 13 & -7 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} +4[2] \\ \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 39 & 0 & -15 & -29 \\ 0 & 13 & -7 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From here one sees that the kernel has dimension 2, and a basis is

$$\{\mathbf{v}_2, \mathbf{v}_1\} = \left\{ \begin{bmatrix} 29 \\ 51 \\ 0 \\ 39 \end{bmatrix}, \begin{bmatrix} 15 \\ 21 \\ 39 \\ 0 \end{bmatrix} \right\}. \tag{4}$$

Thus there are two chains (to EV $\lambda = 1$). Each chain ends on some vector in the span of these two vectors (4). The sum of the lengths of the chains is $n = 4$, thus there are two options: $2, 2$ or $3, 1$.

To continue these chains (and find out their lengths), it is necessary to find out which vectors in the span of (4) lie in the image of $A - Id$.

To do this, we repeat the previous row reduction doing simultaneously the same with $\mathbf{v}_1, \mathbf{v}_2$. (When you write yourself, you don't need to write again the whole matrix reduction, but you should make clear what you are doing!)

We will have to repeat this reduction few times. Now pay only attention to the extra columns '(a)' for $\mathbf{v}_1$ and '(b)' for $\mathbf{v}_2$, which are the two vectors in (4).

$$
\begin{array}{ccccc}
\text{(c)-}\frac{41}{3}\text{(a)} & \text{(c)} & \text{2(a)+3(b)} & \text{(a)} & \text{(b)} \\
-234 & -29 & 117 & 15 & 29 \\
-338 & -51 & 195 & 21 & 51 \\
-533 & 0 & 78 & 39 & 0 \\
0 & 0 & 117 & 0 & 39
\end{array}
\quad
\begin{bmatrix} 156 & -91 & -11 & 3 \\ 192 & -113 & -13 & 5 \\ 597 & -341 & -46 & 2 \\ -150 & 83 & 13 & 3 \end{bmatrix}
\begin{array}{l} +[4] \\ \\ -3[2] \\ \\ \end{array}
\tag{5}
$$

$$
\begin{array}{ccccc}
 & -29 & & 15 & 68 \\
 & -51 & & 21 & 51 \\
 & 153 & & -24 & -153 \\
 & 0 & & 0 & 39
\end{array}
\quad
\begin{bmatrix} 6 & -8 & 2 & 6 \\ 192 & -113 & -13 & 5 \\ 21 & -2 & -7 & -13 \\ -150 & 83 & 13 & 3 \end{bmatrix}
\begin{array}{l} :2 \\ +[4] \\ \\ \\ \end{array}
$$

$$
\begin{array}{ccccc}
 & -29/2 & & 7.5 & 34 \\
 & -51 & & 21 & 90 \\
 & 153 & & -24 & -153 \\
 & 0 & & 0 & 39
\end{array}
\quad
\begin{bmatrix} 3 & -4 & 1 & 3 \\ 42 & -30 & 0 & 8 \\ 21 & -2 & -7 & -13 \\ -150 & 83 & 13 & 3 \end{bmatrix}
\begin{array}{l} \\ -14[1] \\ -7[1] \\ +50[1] \end{array}
$$

$$
\begin{array}{ccccc}
 & -14.5 & & 7.5 & 34 \\
 & 152 & & -84 & -386 \\
 & 254.5 & & -76.5 & -391 \\
 & -725 & & 375 & 1739
\end{array}
\quad
\begin{bmatrix} 3 & -4 & 1 & 3 \\ 0 & 26 & -14 & -34 \\ 0 & 26 & -14 & -34 \\ 0 & -117 & 63 & 153 \end{bmatrix}
\begin{array}{l} \\ :2 \\ -[2] \\ +9/2[2] \end{array}
$$

$$
\begin{array}{ccccc}
-117 & -14.5 & 117 & 7.5 & 34 \\
650 & 76 & -663 & -42 & -193 \\
0 & 102.5 & 0 & 7.5 & -5 \\
0 & -41 & 0 & -3 & 2
\end{array}
\quad
\begin{bmatrix} 3 & -4 & 1 & 3 \\ 0 & 13 & -7 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{array}{l} \cdot 13 \\ \\ \\ \\ \end{array}
\tag{6}
$$

$$
\begin{array}{ccccc}
-1521 & & 1521 & & \\
650 & & -663 & & \\
0 & & 0 & & \\
0 & & 0 & &
\end{array}
\quad
\begin{bmatrix} 39 & -52 & 13 & 39 \\ 0 & 13 & -7 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{array}{l} +4[2] \\ \\ \\ \\ \end{array}
$$

$$
\begin{array}{ccccc}
1079 & & -1131 & & \\
650 & & -663 & & \\
0 & & 0 & & \\
0 & & 0 & &
\end{array}
\quad
\begin{bmatrix} 39 & 0 & -15 & -29 \\ 0 & 13 & -7 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\tag{7}
$$

Look left of the fifth matrix (6). The bottom two rows of the matrix are 0, but the entries in columns (a) and (b) are not. This means that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not in the image of $A - Id$.

If we have two chains of length 2, there would be two linearly independent vectors $\mathbf{v}_{1,2}$ and $\mathbf{v}_{2,2}$ sent to $\mathbf{v}_{1,1}, \mathbf{v}_{2,1}$, which is a basis of span $\{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathbf{v}_1, \mathbf{v}_2$ would both lie in $\mathrm{Im}(A - Id)$, which is not the case. This argument is already enough for us to conclude that the chain lengths are $3, 1$, and thus the Jordan normal form of A is

$$
A' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}
$$

However, we have to work more to find a Jordan basis.

We found that $\dim(\mathrm{Im}(A - Id) \cap \mathrm{Ker}(A - Id)) < 2$. If $\dim(\mathrm{Im}(A - Id) \cap \mathrm{Ker}(A - Id)) = 0$, both chains would break up, which we know is impossible. Thus $\dim(\mathrm{Im}(A - Id) \cap \mathrm{Ker}(A - Id)) = 1$. To continue one of the chains (the one of length 3), we need to find a vector in $\mathrm{Ker}(A - Id) = \mathrm{span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$ which lies in $\mathrm{Im}(A - Id)$.

In (6) we have to find some linear combination of columns (a) and (b) making the two bottom entries to 0. This is seen to be

$$\mathbf{v}_{1,1} = 2(a) + 3(b) = 2\mathbf{v}_1 + 3\mathbf{v}_2 = \begin{bmatrix} 117 \\ 195 \\ 78 \\ 117 \end{bmatrix}. \tag{9}$$

As $\mathbf{v}_{2,1}$ we may take any vector linearly independent from $\mathbf{v}_{1,1}$ in $\mathrm{Ker}(A - Id)$, for example (to keep entries small),

$$\mathbf{v}_{2,1} = (a)/3 = \begin{bmatrix} 5 \\ 7 \\ 13 \\ 0 \end{bmatrix}.$$

The second chain (which is just an eigenvector) is thus done, and we leave it. We continue with the first (longer) chain.

Since row and column operations commute, we need to calculate the linear combination (9) only in (5) (to know what is $2\mathbf{v}_1 + 3\mathbf{v}_2$), and from (6) on, to find a preimage $\mathbf{v}_{1,2}$ (under $A - Id$). Note that this preimage is defined only up to $\mathrm{Ker}(A - Id)$, which we already know. Thus it is enough to find one preimage $\mathbf{v}_3$, and I do this here most simply setting the last two components $(\mathbf{v}_3)_3 = (\mathbf{v}_3)_4 = 0$.

$$\mathbf{v}_3 = (c) = \begin{bmatrix} -1131/39 \\ -663/13 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -29 \\ -51 \\ 0 \\ 0 \end{bmatrix}$$

Again, we cannot be sure to take $\mathbf{v}_3 = \mathbf{v}_{1,2}$, because we need that $\mathbf{v}_{1,2}$ lies in $\mathrm{Im}(A - Id)$ (to have $\mathbf{v}_{1,3}$).

We row-reduce $\mathbf{v}_3$ again until (6). The last two entries are not zero, meaning $\mathbf{v}_3$ lies outside of $\mathrm{Im}(A - Id)$. However, we know that we can correct $\mathbf{v}_3$ by some element in $\mathrm{Ker}(A - Id)$ to lie in $\mathrm{Im}(A - Id)$ (i.e., the last two entries to become 0). One possible correction is seen to be

$$\mathbf{v}_{1,2} = (c) - \frac{41}{3}(a) = \begin{bmatrix} -234 \\ -338 \\ -533 \\ 0 \end{bmatrix},$$

again because this linear combination makes the last two entries in (6) to be 0. In (7) this linear combination leads (again setting the last two components to 0) to the preimage

$$\mathbf{v}_{1,3} = \begin{bmatrix} 83/3 \\ 50 \\ 0 \\ 0 \end{bmatrix}.$$

This is the final element of the chain, and thus no correction by elements in $\mathrm{Ker}(A - Id)$ is necessary (we seek no further preimage).

Thus now we got our Jordan basis β together.

$$\beta = \{\mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \mathbf{v}_{1,3}, \mathbf{v}_{2,1}\} = \left\{ \begin{bmatrix} 117 \\ 195 \\ 78 \\ 117 \end{bmatrix}, \begin{bmatrix} -234 \\ -338 \\ -533 \\ 0 \end{bmatrix}, \begin{bmatrix} 83/3 \\ 50 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 13 \\ 0 \end{bmatrix} \right\}.$$

Verify

$$(A - Id)(\mathbf{v}_{i,1}) = 0, \quad (A - Id)(\mathbf{v}_{1,3}) = \mathbf{v}_{1,2} \text{ and } \quad (A - Id)(\mathbf{v}_{1,2}) = \mathbf{v}_{1,1}. \tag{10}$$

One can check that β is a basis, but it is more elegant to argue formally from (10) without further numeric calculation.

We know $\mathbf{v}_{1,1}$, $\mathbf{v}_{2,1}$ are linearly independent. Also $(A - Id)(\mathbf{v}_{1,2}) = \mathbf{v}_{1,1} \neq 0$, so $\mathbf{v}_{1,2} \notin \operatorname{Ker}(A - Id) = \operatorname{span}\{\mathbf{v}_{1,1}, \mathbf{v}_{2,1}\}$. Thus $\mathbf{v}_{1,2}$ is linearly independent from $\mathbf{v}_{1,1}$, $\mathbf{v}_{2,1}$. Finally $(A - Id)(\operatorname{span}\{\mathbf{v}_{1,2}, \mathbf{v}_{1,1}, \mathbf{v}_{2,1}\}) = \operatorname{span}(\{\mathbf{v}_{1,1}\})$, but $(A - Id)(\mathbf{v}_{1,3}) = \mathbf{v}_{1,2} \notin \operatorname{span}(\{\mathbf{v}_{1,1}\})$. This shows that $\mathbf{v}_{1,3}$ is linearly independent from $\beta \setminus \{\mathbf{v}_{1,3}\}$, and thus indeed β is a basis.

And in this basis β we have $[L_A]_\beta = A'$ in (8), as we liked. □

Problem 2. Find the Jordan normal form and a Jordan basis of the matrix

$$A = \begin{bmatrix} -80 & 44 & 8 & 4 \\ -115 & 64 & 11 & 5 \\ -220 & 116 & 25 & 15 \\ 24 & -12 & -3 & -1 \end{bmatrix}.$$

In particular, determine all eigenvalues, and argue that your Jordan basis is indeed a basis.

Hint: χ_A splits and all eigenvalues are integers. Think about how to use this information to determine the eigenvalues more easily.

Solution. First, we determine the eigenvalues. One can determine the characteristic polynomial. Rather, we calculate the determinant. We find

$$\det A = 16\,.$$

I do row operations. Let us fix the following notation.

- An entry '$\leftrightarrow [m]$' right to a row of a matrix means that in the following matrix, the row is exchanged with row m (type I operation)
- An entry '$\cdot\mu$' or '$:\mu$' right to a row of a matrix means that in the following matrix, this row is multiplied, resp. divided by μ (type II operation).
- An entry '$+\mu[m]$' right to a row of a matrix means that in the following matrix, μ times row m of the current matrix is added (type III operation)

$$\begin{vmatrix} -80 & 44 & 8 & 4 \\ -115 & 64 & 11 & 5 \\ -220 & 116 & 25 & 15 \\ 24 & -12 & -3 & -1 \end{vmatrix} \begin{matrix} :4 \\ \\ \\ \\ \end{matrix} = 4 \cdot \begin{vmatrix} -20 & 11 & 2 & 1 \\ -115 & 64 & 11 & 5 \\ -220 & 116 & 25 & 15 \\ 24 & -12 & -3 & -1 \end{vmatrix} \begin{matrix} \\ -5[1] \\ 15[1] \\ +[1] \end{matrix}$$

$$= 4 \cdot \begin{vmatrix} -20 & 11 & 2 & 1 \\ -15 & 9 & 1 & 0 \\ 80 & -49 & -5 & 0 \\ 4 & -1 & -1 & 0 \end{vmatrix}$$

$$= -4 \cdot \begin{vmatrix} -15 & 9 & 1 \\ 80 & -49 & -5 \\ 4 & -1 & -1 \end{vmatrix} \begin{matrix} \\ +5[1] \\ +[1] \end{matrix}$$

$$= -4 \cdot \begin{vmatrix} -15 & 9 & 1 \\ 5 & -4 & 0 \\ -11 & 8 & 0 \end{vmatrix}$$

$$= -4 \cdot \begin{vmatrix} 5 & -4 \\ -11 & 8 \end{vmatrix} = -4 \cdot (40 - 44) = 16\,.$$

With the hint that χ_A splits and all λ are integers, we have

$$\chi_A(t) = \prod_{i=1}^{t} (t - \lambda_i) \tag{11}$$

and thus $\lambda_1\lambda_2\lambda_3\lambda_4 = 16$ with $\lambda_i \in \mathbb{Z}$. Then all $\lambda_i \mid 16$.

We test that $A \pm Id$ is invertible, by calculation of the determinant.

$$\chi_A(1) = \det(A - Id) = 1\,, \qquad \chi_A(-1) = \det(A + Id) = 81\,. \tag{12}$$

The determinant $\det(A - Id)$ was in Quiz 6.

We have thus that no $\lambda_i = \pm 1$. Then we can conclude $\lambda_i = \pm 2$, and don't need to test eigenvalues $\lambda_i = \pm 4, \pm 8, \pm 16$. If you don't see this, you can test 6 more 4×4 matrices. (The rule applies here again: *either you think a little, or calculate a lot.*)

One can test that $\lambda = -2$ is not an eigenvalue (one more determinant to calculate), but it can be seen (saving this calculation) from $\chi_A(1) = 1$ (12) that in (11) all factors must evaluate to ± 1 in $t = 1$, thus must be t or $t - 2$ (and none can be $t + 2$). Thus

$$\chi_A(t) = (t - 2)^4\,.$$

We see thus that $\lambda = -2$ is not an eigenvalue, and we have all $\lambda_i = 2$.

We consider thus $\lambda = 2$.

Next, we find the eigenspace, which is the kernel of $A - \lambda Id = A - 2Id$. We do row reduction. Let us fix the following notation.

- An entry '$\to [m]$' right to a row of a matrix means that in the following matrix, the row is taken out and inserted as row m (type I operations)
- An entry '$\cdot\mu$' or '$:\mu$' right to a row of a matrix means that in the following matrix, this row is multiplied, resp. divided by μ (type II operation).
- An entry '$+\mu[m]$' right to a row of a matrix means that in the following matrix, μ times row m of the current matrix is added (type III operation)

Again we explain the columns added on the left later. Some zero entries in the matrices (but not columns left of them!) are omitted.

$$
\begin{array}{rrrr}
(a) & (b) & (c) & (d) \\
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
25 & -7 & -5 & 1 \\
-9 & 3 & 11 & -2
\end{array}
\quad
\begin{array}{c}
\\
\left[\begin{array}{rrrr}
-82 & 44 & 8 & 4 \\
-115 & 62 & 11 & 5 \\
-220 & 116 & 23 & 15 \\
24 & -12 & -3 & -3
\end{array}\right]
\end{array}
\quad
\begin{array}{l}
\\
:2 \\
\\
\\
:3
\end{array}
\tag{13}
$$

$$
\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
25 & -7 & -5 & 1 \\
-3 & 1 & 11/3 & -2/3
\end{array}
\quad
\left[\begin{array}{rrrr}
-41 & 22 & 4 & 2 \\
-115 & 62 & 11 & 5 \\
-220 & 116 & 23 & 15 \\
8 & -4 & -1 & -1
\end{array}\right]
\quad
\begin{array}{l}
+2[4] \\
+5[4] \\
+15[4] \\
\\
\end{array}
$$

$$
\begin{array}{rrrr}
-5 & 2 & 22/3 & -4/3 \\
-15 & 6 & 55/3 & -10/3 \\
-20 & 8 & 50 & -9 \\
-3 & 1 & 11/3 & -2/3
\end{array}
\quad
\left[\begin{array}{rrrr}
-25 & 14 & 2 & 0 \\
-75 & 42 & 6 & 0 \\
-100 & 56 & 8 & 0 \\
8 & -4 & -1 & -1
\end{array}\right]
\quad
\begin{array}{l}
\to [3] \\
-3[1] \\
-4[1] \\
\\
\end{array}
$$

$$
\begin{array}{rrrr}
0 & 0 & -11/3 & 2/3 \\
0 & 0 & 62/3 & -11/3 \\
-5 & 2 & 22/3 & -4/3 \\
-3 & 1 & 11/3 & -2/3
\end{array}
\quad
\left[\begin{array}{rrrr}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-25 & 14 & 2 & 0 \\
8 & -4 & -1 & -1
\end{array}\right]
\quad
\begin{array}{l}
\\
\\
:2 \\
\cdot(-1)
\end{array}
\tag{14}
$$

$$
\begin{array}{rrrr}
0 & 0 & -11/3 & 2/3 \\
0 & 0 & 62/3 & -11/3 \\
-5/2 & 1 & 11/3 & -2/3 \\
3 & -1 & -11/3 & 2/3
\end{array}
\quad
\left[\begin{array}{rrrr}
 & & & \\
 & & & \\
-12.5 & 7 & 1 & \\
-8 & 4 & 1 & 1
\end{array}\right]
\quad
\begin{array}{l}
\\
\\
\\
-[3]
\end{array}
$$

$$
\begin{array}{rr}
0 & 0 \\
0 & 0 \\
-5/2 & 1 \\
11/2 & -2
\end{array}
\quad
\left[\begin{array}{rrrr}
 & & & \\
 & & & \\
-12.5 & 7 & 1 & \\
4.5 & -3 & 0 & 1
\end{array}\right]
\tag{15}
$$

From here one sees that the kernel has dimension 2, and a basis is

$$
\{\mathbf{v}_2, \mathbf{v}_1\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 25 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -7 \\ 3 \end{bmatrix} \right\}. \tag{16}
$$

Thus there are two chains (to EV $\lambda = 2$). Each chain ends on some vector in the span of these two vectors (16). The sum of the lengths of the chains is $n = 4$, thus there are two options: $2, 2$ or $3, 1$.

To continue these chains (and find out their lengths), it is necessary to find out which vectors in the span of (16) lie in the image of $A - 2Id$.

To do this, we repeat the previous row reduction doing simultaneously the same with $\mathbf{v}_1, \mathbf{v}_2$.

We will have to repeat this reduction few times. Now pay attention to the extra columns '(a)' for $\mathbf{v}_1$ and '(b)' for $\mathbf{v}_2$, which are the two vectors in (16).

Look left of the fifth matrix (14). The top two rows of the matrix are 0, and the entries in columns (a) and (b) are also. This means that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the image of $A - 2Id$.

Thus there are vectors $\mathbf{v}_{i,2}$ with $(A-2Id)\mathbf{v}_{i,2}=\mathbf{v}_{i,1}$ for $i=1,2$. This means that we have two chains of length 2. This argument is already enough for us to conclude that the chain lengths are $2,2$, and thus the Jordan normal form of A is

$$A' = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{17}$$

However, we have to work a bit more to find a Jordan basis. We can take

$$\mathbf{v}_{1,1} = (a) = \begin{bmatrix} 2 \\ 0 \\ 25 \\ -9 \end{bmatrix}, \mathbf{v}_{2,1} = (b) = \begin{bmatrix} 0 \\ 1 \\ -7 \\ 3 \end{bmatrix}$$

to be the end (eigen)vectors of the chains.

We found that $\dim(\mathrm{Im}(A-2Id)\cap\mathrm{Ker}(A-2Id))=2$. In (14) we have to find some vectors $\mathbf{v}_{i,2}$ with $(A-2Id)\mathbf{v}_{i,2}=\mathbf{v}_{i,1}$.

We need to continue calculating the row reduction from (14) on, to find preimages $\mathbf{v}_{i,2}$ (under $A-2Id$). Note that these preimages are defined only up to $\mathrm{Ker}(A-2Id)$. But it is enough to find one preimage $\mathbf{v}_{i,2}$, and I do this here most simply setting the first two components $(\mathbf{v}_{i,2})_1=(\mathbf{v}_{i,2})_2=0$.

$$\mathbf{v}_{1,2} = (c)/2 = \begin{bmatrix} 0 \\ 0 \\ -5/2 \\ -11/2 \end{bmatrix}, \quad \mathbf{v}_{2,2} = (d) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}.$$

These are the final elements of the chains, and thus no correction by elements in $\mathrm{Ker}(A-2Id)$ is necessary (we seek no further preimages).

To demonstrate that indeed the chains break up at $\mathbf{v}_{i,2}$, I have once more row reduced $\mathbf{v}_{i,2}$ as columns (c), (d). In (14) one sees that the determinant of the two left top rows/columns is

$$\begin{vmatrix} -11/3 & 2/3 \\ 62/3 & -11/3 \end{vmatrix} = \frac{1}{3} \neq 0.$$

This means that no non-trivial linear combination, even after adding elements in $\mathrm{Ker}(A-2Id)$, lies in the image of $A-2Id$. This displays that indeed the chains break up.

Thus now we got our Jordan basis β together. To remove the fraction, I multiply the first chain by 2.

$$\beta = \{2\mathbf{v}_{1,1}, 2\mathbf{v}_{1,2}, \mathbf{v}_{2,1}, \mathbf{v}_{2,2}\} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 50 \\ -18 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -5 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

Verify that $(A-2Id)(\mathbf{v}_{i,1})=0$ and $(A-2Id)(\mathbf{v}_{i,2})=\mathbf{v}_{i,1}$.

$$\begin{bmatrix} -80 & 44 & 8 & 4 \\ -115 & 64 & 11 & 5 \\ -220 & 116 & 25 & 15 \\ 24 & -12 & -3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 50 & -5 & -7 & 1 \\ -18 & 11 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 100 & 40 & -14 & -5 \\ -36 & 4 & 6 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 50 & -5 & -7 & 1 \\ -18 & 11 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

One can check that β is a basis, but it is better to argue formally without numeric calculation.

We know $\mathbf{v}_{1,1}$, $\mathbf{v}_{2,1}$ are linearly independent. Also $(A-2Id)(\mathbf{v}_{i,2}) = \mathbf{v}_{i,1} \neq 0$, so $\mathbf{v}_{i,2} \notin \operatorname{Ker}(A-2Id) = \operatorname{span}\{\mathbf{v}_{1,1}, \mathbf{v}_{2,1}\}$. Thus $\mathbf{v}_{i,2}$ are linearly independent from $\mathbf{v}_{1,1}$, $\mathbf{v}_{2,1}$. Finally $(A-2Id)(\operatorname{span}\{\mathbf{v}_{1,2}, \mathbf{v}_{1,1}, \mathbf{v}_{2,1}\}) = \operatorname{span}(\{\mathbf{v}_{1,1}\})$, but $(A-2Id)(\mathbf{v}_{2,2}) = \mathbf{v}_{2,1} \notin \operatorname{span}(\{\mathbf{v}_{1,1}\})$. This shows that $\mathbf{v}_{2,2}$ is linearly independent from $\beta \setminus \{\mathbf{v}_{2,2}\}$, and thus indeed β is a basis.

And in this basis β we have $[L_A]_\beta = A'$ in (17), as we liked. □

Problem 3. Find the Jordan normal form and a Jordan basis of the matrix

$$A = \begin{bmatrix} 28 & -14 & -4 & -3 \\ 34 & -17 & -5 & -4 \\ 115 & -58 & -15 & -9 \\ -19 & 10 & 2 & 0 \end{bmatrix}.$$

Hint: χ_A splits and all eigenvalues are integers.

Solution. First, one must find the eigenvalues. One can determine the characteristic polynomial. But more easily calculate $\det A = 1$, implying that all EV must be ± 1 (when they are all integers and χ_A splits). Testing $\lambda = 1$ shows that it is not an eigenvalue, thus $\lambda = -1$, and we find

$$\chi(t) = t^4 + 4t^3 + 6t^2 + 4t + 1\,.$$

($\lambda = -1$ is the only eigenvalue).

Next, we find the eigenspace, which is the kernel of $A - \lambda Id = A + Id$. We do row reduction. (I don't indicate the row operations.)

Again we explain the columns added on the left later.

$$\begin{array}{cccccc} \text{(a)} & \text{(b)} & 2\text{(a)}+3\text{(b)} & \text{(c)} & 2\text{(c)}+\text{(b)}=\text{(d)} & \text{(e)} \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 14 & -8 & 4 & 1 & -6 & 3 \\ -9 & 6 & 0 & -2 & 2 & -4 \end{array} \quad \begin{bmatrix} 29 & -14 & -4 & -3 \\ 34 & -16 & -5 & -4 \\ 115 & -58 & -14 & -9 \\ -19 & 10 & 2 & 1 \end{bmatrix} \tag{18}$$

$$\begin{array}{cccccc} -26 & 18 & 2 & -6 & 6 & -12 \\ -36 & 25 & 3 & -8 & 9 & -16 \\ -67 & 46 & 4 & -17 & 12 & -33 \\ -9 & 6 & 0 & -2 & 2 & -4 \end{array} \quad \begin{bmatrix} -28 & 16 & 2 & 0 \\ -42 & 24 & 3 & 0 \\ -56 & 32 & 4 & 0 \\ -19 & 10 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{cccccc} 3 & -2 & 0 & 1 & 0 & 2 \\ -15 & 10 & 0 & -5 & 0 & -9 \\ -13 & 9 & 1 & -3 & 3 & -6 \\ -9 & 6 & 0 & -2 & 2 & -4 \end{array} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -14 & 8 & 1 & 0 \\ -19 & 10 & 2 & 1 \end{bmatrix} \tag{19}$$

$$\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 3 \\ -2 & -4 \end{array} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -14 & 8 & 1 & 0 \\ 9 & -6 & 0 & 1 \end{bmatrix} \tag{20}$$

From here one sees that the kernel has dimension 2, and a basis is

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 14 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -8 \\ 6 \end{bmatrix} \right\}. \tag{21}$$

Thus there are two chains (to EV $\lambda = 1$). Each chain ends on some vector in the span of these two vectors (21). The sum of the lengths of the chains is $n = 4$, thus there are two options: $2, 2$ or $3, 1$.

To continue these chains (and find out their lengths), it is necessary to find out which vectors in the span of (21) lie in the image of $A + Id$.

To do this, we repeat the previous row reduction doing simultaneously the same with $\mathbf{v}_1, \mathbf{v}_2$. (When you write yourself, you don't need to write again the whole matrix reduction, but you should make clear what you are doing!)

We will have to repeat this reduction few times. Now pay only attention to the extra columns '(a)' for $\mathbf{v}_1$ and '(b)' for $\mathbf{v}_2$, which are the two vectors in (21).

Look left of the third matrix (19). The top two rows of the matrix are 0, but the entries in columns (a) and (b) are not. This means that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not in the image of $A + Id$.

If we have two chains of length 2, there would be two linearly independent vectors $\mathbf{v}_{1,2}$ and $\mathbf{v}_{2,2}$ sent to $\mathbf{v}_{1,1}, \mathbf{v}_{2,1}$, which is a basis of span $\{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathbf{v}_1, \mathbf{v}_2$ would both lie in $\mathrm{Im}(A+Id)$, which is not the case. This argument is already enough for us to conclude that the chain lengths are $3, 1$, and thus the Jordan normal form of A is

$$A' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{22}$$

However, we have to work more to find a Jordan basis.

We found that $\dim(\mathrm{Im}(A + Id) \cap \mathrm{Ker}(A + Id)) < 2$. If $\dim(\mathrm{Im}(A + Id) \cap \mathrm{Ker}(A + Id)) = 0$, both chains would break up, which we know is impossible. Thus $\dim(\mathrm{Im}(A+Id) \cap \mathrm{Ker}(A+Id)) = 1$. To continue one of the chains (the one of length 3), we need to find a vector in $\mathrm{Ker}(A + Id) = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ which lies in $\mathrm{Im}(A + Id)$.

In (19) we have to find some linear combination of columns (a) and (b) making the two top entries to 0. This is seen to be

$$\mathbf{v}_{1,1} = 2(a) + 3(b) = 2\mathbf{v}_1 + 3\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 0 \end{bmatrix}. \tag{23}$$

As $\mathbf{v}_{2,1}$ we may take any vector linearly independent from $\mathbf{v}_{1,1}$ in $\mathrm{Ker}(A + Id)$, for example (to keep entries small),

$$\mathbf{v}_{2,1} = (b) = \begin{bmatrix} 0 \\ 1 \\ -8 \\ 6 \end{bmatrix}.$$

The second chain (which is just an eigenvector) is thus done, and we leave it. We continue with the first (longer) chain.

Since row and column operations commute, we need to calculate the linear combination (23) only in (18) (to know what is $2\mathbf{v}_1 + 3\mathbf{v}_2$), and from (19) on, to find a preimage $\mathbf{v}_{1,2}$ (under $A + Id$). Note that this preimage is defined only up to $\mathrm{Ker}(A + Id)$, which we already know. Thus it is enough to find one preimage $\mathbf{v}_3$, and I do this here most simply setting the first two components $(\mathbf{v}_3)_1 = (\mathbf{v}_3)_2 = 0$.

$$\mathbf{v}_3 = (c) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

Again, we cannot be sure to take $\mathbf{v}_3 = \mathbf{v}_{1,2}$, because we need that $\mathbf{v}_{1,2}$ lies in $\mathrm{Im}(A + Id)$ (to have $\mathbf{v}_{1,3}$).

We row-reduce $\mathbf{v}_3$ again until (19). The first two entries are not zero, meaning $\mathbf{v}_3$ lies outside of $\mathrm{Im}(A + Id)$. However, we know that we can correct $\mathbf{v}_3$ by some element in $\mathrm{Ker}(A + Id)$ to lie in $\mathrm{Im}(A + Id)$ (i.e., the first two entries to become 0). One possible correction is seen to be

$$2\mathbf{v}_{1,2} = 2(c) + (b) = (d) = \begin{bmatrix} 0 \\ 1 \\ -6 \\ 2 \end{bmatrix},$$

again because this linear combination makes the first two entries in (19) to be 0. (The factor 2 is added to keep the entries to be integers.) In (20) this linear combination leads (again setting the last two components to 0) to the preimage

$$2\mathbf{v}_{1,3} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -4 \end{bmatrix}.$$

This is the final element of the chain, and thus no correction by elements in $\mathrm{Ker}(A+Id)$ is necessary (we seek no further preimage). The last column left of the matrices is the test that indeed no multiple $\mathbf{v}_{1,3}$ can be corrected by an element in $\mathrm{Ker}(A + Id)$ to lie in $\mathrm{Im}(A + Id)$: the first two rows of column (e) in (19) are not a multiple of the first two rows of column (a) or (b) (which are multiples of each other, as we found out before).

Thus now we got our Jordan basis β together.

$$\beta = \{\mathbf{v}_{2,1}, 2\mathbf{v}_{1,1}, 2\mathbf{v}_{1,2}, 2\mathbf{v}_{1,3}\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -4 \end{bmatrix} \right\}.$$

Verify

$$(A + Id)(\mathbf{v}_{i,1}) = 0, \quad (A + Id)(\mathbf{v}_{1,3}) = \mathbf{v}_{1,2} \text{ and } \quad (A + Id)(\mathbf{v}_{1,2}) = \mathbf{v}_{1,1}, \tag{24}$$

by

$$\begin{bmatrix} 28 & -14 & -4 & -3 \\ 34 & -17 & -5 & -4 \\ 115 & -58 & -15 & -9 \\ -19 & 10 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ -8 & 8 & -6 & 3 \\ 6 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 4 & 0 \\ -1 & -6 & 5 & 1 \\ 8 & -8 & 14 & -9 \\ -6 & 0 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ -8 & 8 & -6 & 3 \\ 6 & 0 & 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

One can check that β is a basis, but it is more elegant to argue formally from (24) without further numeric calculation.

We know $\mathbf{v}_{1,1}$, $\mathbf{v}_{2,1}$ are linearly independent. Also $(A + Id)(\mathbf{v}_{1,2}) = \mathbf{v}_{1,1} \neq 0$, so $\mathbf{v}_{1,2} \notin \mathrm{Ker}(A + Id) = \mathrm{span}\,\{\mathbf{v}_{1,1}, \mathbf{v}_{2,1}\}$. Thus $\mathbf{v}_{1,2}$ is linearly independent from $\mathbf{v}_{1,1}$, $\mathbf{v}_{2,1}$. Finally $(A + Id)(\mathrm{span}\,\{\mathbf{v}_{1,2}, \mathbf{v}_{1,1}, \mathbf{v}_{2,1}\}) = \mathrm{span}\,(\{\mathbf{v}_{1,1}\})$, but $(A + Id)(\mathbf{v}_{1,3}) = \mathbf{v}_{1,2} \notin \mathrm{span}\,(\{\mathbf{v}_{1,1}\})$. This shows that $\mathbf{v}_{1,3}$ is linearly independent from $\beta \setminus \{\mathbf{v}_{1,3}\}$, and thus indeed β is a basis.

And in this basis β we have $[L_A]_\beta = A'$ in (22), as we liked. $\square$