7. Top spaces and cont fct's

1.1 Top Spaces, top. operations, separation

<u>Def.</u> X set $\mathcal{A} \subset \mathcal{P}(X)$ <u>topology</u> on X if

(1) $\emptyset, X \in \mathcal{A}$

(2) closed under arb. union

$$\mathcal{A}' \subset \mathcal{A} \qquad \bigcup \mathcal{A}' = \bigcup_{A' \in \mathcal{A}'} A' \in \mathcal{A}$$

(3) closed under <u>simple</u> $\cap$

$$A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$$

<u>Rem:</u> $(3) \Longleftrightarrow \forall \mathcal{A}' \subset \mathcal{A} \quad |\mathcal{A}'| < \infty \quad \bigcap \mathcal{A}' \in \mathcal{A}$

closed under <u>finite</u> $\cap$

<u>Def:</u> $(X, \mathcal{A})$ top call $A \in \mathcal{A}$ an <u>open</u> set

<u>Ex.</u> $\mathcal{A} = \mathcal{P}(X)$ discrete topology write $(X, \mathcal{P}(X)) = X_{discr}$

$\mathcal{A} = \{\emptyset, X\}$ trivial topology (indiscrete)

<u>Ex</u> $\mathcal{A}$ = maximal chain in $(\mathcal{P}(X), \subseteq) \to$ HW.

Ex. $\mathcal{A} = \{ A \subset X : |X \setminus A| < \infty \} \cup \{\emptyset\}$

finite complement topology

triv top. motivates

Def: $(X, \mathcal{A})$ top. space

$x_1, x_2 \in X$ are call top. indistinguishable

if $\forall A \in \mathcal{A} \quad (x_1 \in A \iff x_2 \in A)$

means

$x_1, x_2 \in A$ or $x_1, x_2 \notin A$

top. $\mathcal{A}$ cannot dist. x_1, x_2

Def: (T_0 Kholmogorov axiom) very basic separation axiom

$\mathcal{A}$ is T_0 if $\nexists$ top. indistinguishable pts

$\iff \forall x_1 \neq x_2 \quad \exists A \in \mathcal{A} \quad x_1 \in A, x_2 \notin A$ or $x_1 \notin A, x_2 \in A$

Let $\mathcal{A}$ be a top. on X. Define an equivalence rel. on X by

$x_1 \sim x_2 \iff x_1$ is top. indistinguishable fr x_2

Then let

eq. class of x under $\sim$

$$\widetilde{X} = X/_\sim = \{ [x]_\sim : x \in X \}$$

and define a top. $\widetilde{\mathcal{A}}$ on $\widetilde{X}$ by

$$\widetilde{\mathcal{A}} = \{ \{ [x]_\sim : x \in A \} : A \in \mathcal{A} \}$$

this top. identifies (and removes) top. indistinguishable pts.

$(\widetilde{X}, \widetilde{\mathcal{A}})$ is called Kolmogorov quotient of $(X, \mathcal{A})$

$\widetilde{X}$ is T_0

Ex. If $\mathcal{A} = \{\emptyset, X\}$ then $\widetilde{X} = \{*\}$

triv. top. is not T_0 unless $|X| = 1$

often will assume T_0

$\{A_i\}_{i \in I}$ closed $\Rightarrow \bigcap_{i \in I} A_i$

A, B closed $A \cup B$ closed

Def: $(X, \mathcal{A})$ $A \subset X$ is called closed if $X \setminus A$ is open

Def: $x \in A$ an $O \in \mathcal{A}$ $O \ni x$ is called neighborhood of x

Def: $(X, \mathcal{A})$ $A \subset X$ any set

let $\mathrm{Int}(A) = \bigcup \{ A' \in \mathcal{A} : A' \subset A \}$

the interior of A $= \{ x \in X : \exists\, O \in \mathcal{A} \;\; x \in O \subset A \}$ $(x \in A)$

interior point

Lemma. 1) $\mathrm{Int}(A) \subset A$

2) $\mathrm{Int}(A) = A \iff A$ is open $\left(\Rightarrow \mathrm{Int}(\emptyset) = \emptyset,\ \mathrm{Int}(X) = X \right)$

$Int(Int(A)) = Int(A)$. ($\Leftrightarrow$ $Int(A)$ open)

Pf. 1) ✓

2) "$\Leftarrow$" A open $\quad A \in \mathcal{A}' = \{A' \in \mathcal{A} : A' \subset A\}$

$$A \overset{!}{\supset} Int(A) = \bigcup \mathcal{A}' \supset A$$

$$\Rightarrow Int(A) = A.$$

"$\Rightarrow$" $Int(A) = A \qquad A = \bigcup \underbrace{\{A' \in \mathcal{A} : A' \subset A\}}_{\mathcal{A}'}$

$$\mathcal{A}' \subset \mathcal{A} \overset{\text{top. prop.}}{\Longrightarrow} A \in \mathcal{A} \Rightarrow A \text{ open}$$

3) $A \subset B \quad \mathcal{A}' = \{A' \in \mathcal{A} : A' \subset A\}$

$\subset \mathcal{B}' = \{B' \in \mathcal{A} : B' \subset B\}$

$$Int(B) = \bigcup \mathcal{B}' \supset \bigcup \mathcal{A}' = Int(A)$$

4) $Int(A) = \bigcup \mathcal{A}'$ for $\mathcal{A}' \subset \mathcal{A} \Rightarrow$

$Int(A)$ open

$$\overset{2)}{\Longrightarrow} Int(Int(A)) = Int(A).$$

Def: $(X, \mathcal{A})$ $A \subset X$ define the exterior

$$Ext(A) = Int(X \setminus A)$$

try not to use much

$$= \{x \in X : \exists O \in \mathcal{A} \quad O \cap A = \emptyset,\ x \in O\}$$

$x \in Ext(A)$ exterior point.

closure

$$\bar{A} = X \setminus \mathrm{Ext}(A) = X \setminus \mathrm{Int}(X \setminus A)$$

$$= \{ x \in X : \forall O \in \mathcal{A} \quad O \ni x \Rightarrow O \cap A \neq \emptyset \}$$

Lemma: Properties of the closure

$\bar{A} \supset A \qquad \bar{A} = A \Leftrightarrow A$ closed $\quad (\Rightarrow \bar{\emptyset} = \emptyset,\ \bar{X} = X)$

$\bar{\bar{A}} = \bar{A}$ (i.e. $\bar{A}$ is closed)

$A \subset B \Rightarrow \bar{A} \subset \bar{B}$

Rem: V VS $\quad S \subset V$ set $\quad$ span has similar prop:

span$(S) \supset S$

$S \supset S' \Rightarrow$ span$(S) \supset$ span(S')

span(span(S)) = span(S)

operations of this sort are called hull operations

[similar conv(X) convex hull]

so closure is a hull operation

Def: $A \subset X \qquad \mathrm{Bd}\, A = \bar{A} \cap \overline{X \setminus A} = \bar{A} \setminus \mathrm{Int}(A)$

is called (top.) boundary of A

elsewhere

$(\partial A!)$ $\qquad \mathrm{Bd}\, A = \{ x \in X \quad \forall O \in \mathcal{A} \quad O \ni x$

$O \not\subset A \quad O \cap A \neq \emptyset \}$

$x \in \mathrm{Bd}\, A$ is called

Ex. $\mathcal{A}$ = discrete top. all $A \subset X$ open
$\Rightarrow$ all closed

$$\bar{A} = \mathrm{Int}(A) = A \qquad \mathrm{Bd}\, A = \bar{A} \setminus \mathrm{Int}(A) = \emptyset$$

Ex. $\mathcal{A}$ = triv. top. $A \subset X$ $A \neq \emptyset, X$

$$\mathrm{Int}(A) = \emptyset \qquad \bar{A} = X \qquad \mathrm{Bd}\, A = X$$

Def. We say $(X, \mathcal{A})$ is T_1 (Fréchet)
if $\forall x_1 \neq x_2$ $\exists O_1 \ni x_2$ $O_1 \not\ni x_1$ **and**
$O_2 \ni x_2$ $O_2 \not\ni x_1$

$O_{1,2} \in \mathcal{A}$

Lemma $T_1 \overset{(1)}{\Longleftrightarrow}$ points are closed $\overset{(2)}{\Longleftrightarrow}$ finite sets are closed
$\overline{\{x\}} = \{x\}$

Pf. (2) bec. closed is invariant under finite union

(1) $\mathcal{A}$ is T_1 let $x_1 \in X$ $x_2 \neq x_1$

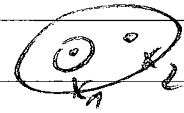
$\exists x_2 \ni O_2$ open $O_2 \not\ni x_1$
$O_2 \cap \{x_1\} = \emptyset$

$\Rightarrow x_2 \notin \overline{\{x_1\}}$ $\forall x_2 \neq x_1$

$\Rightarrow \overline{\{x_1\}} = \{x_1\}$. $\square$

$T_1 \Rightarrow T_0$ but not converse

Ex. $X = \{1, 2\}$



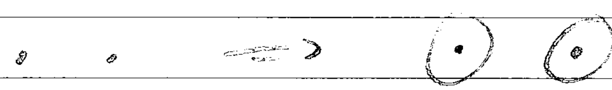
$\mathcal{A}$ is T_0 $\quad \overline{\{x_1\}} = \{x_1, x_2\}$.

Ex. finite comp. top. is T_1
(f.c.)

Def: $(X, \mathcal{A})$ is T_2 (Hausdorff)

if $\forall x_1, x_2 \in X \quad x_1 \neq x_2 \quad \exists O_1 \ni x_1, \; O_2 \ni x_2 \quad O_i$ open

$O_1 \cap O_2 = \emptyset$



$T_2 \Rightarrow T_1 (\Rightarrow T_0)$ but not converse

Ex. $|X| = \infty \quad \mathcal{A} = $ f.c. top on X

$x_1, x_2 \in X \quad x_1 \neq x_2$

$O_1 \ni x_1 \quad O_2 \ni x_2 \quad O_i \in \mathcal{A}$.

$O_1, O_2 \neq \emptyset \Rightarrow |X \setminus O_1|, |X \setminus O_2| < \infty$

$|X \setminus (O_1 \cap O_2)| = |(X \setminus O_1) \cup (X \setminus O_2)| < \infty$

$|X| = \infty \Rightarrow O_1 \cap O_2 \neq \emptyset \quad$ f.c. top is not Hausdorff

Def. $(X,\mathcal{A}')$, $(X,\mathcal{A})$ two top. on same space

we say $\mathcal{A}'$ is finer than $\mathcal{A}$ if $\mathcal{A}' \supset \mathcal{A}$

strictly f. $\mathcal{A}' \supsetneq \mathcal{A}$

coarser $\mathcal{A}' \subset \mathcal{A}$

strictly coarser $\mathcal{A}' \subsetneq \mathcal{A}$

! to usage of "weaker/sharper", "bigger/smaller" elsewhere (try to avoid)

Rem if coarser of $\mathcal{A}$ is T_i $(i=0,1,2)$ $\mathcal{A}'$ finer $\Rightarrow$ $\mathcal{A}'$ is T_i, too)

Rem $\mathcal{A}' \supset \mathcal{A}$ $A \subset X$ $\overline{A}^{\mathcal{A}'} \subset \overline{A}^{\mathcal{A}}$ $\mathrm{Int}_{\mathcal{A}'}(A) \supset \mathrm{Int}_{\mathcal{A}}(A)$ Def: dense, separable

§13 Basis of top.

most important top. we will deal with can be defined through a basis.

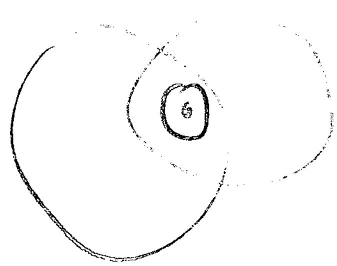
Def. X set $\mathcal{B} \subset \mathcal{P}(X)$ is a basis for top. on X

if

B1) $\forall x \in X \ \exists B \in \mathcal{B} \ x \in B$ $\Leftrightarrow \bigcup \mathcal{B} = X$

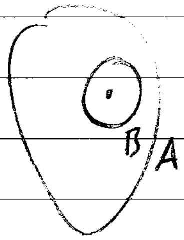
B2) $\forall B_1, B_2 \in \mathcal{B} \ \forall x \in X$

$x \in B_1 \cap B_2 \ \exists B_3 \in \mathcal{B} \ x \in B_3 \subset B_1 \cap B_2$



Df: $(X, \mathcal{B})$ basis of top.

the top. $\mathcal{A}$ generated by X $\qquad \mathcal{A} = \mathcal{A}(\mathcal{B})$



$A \in \mathcal{A} \iff \forall x \in A \ \ \exists B \in \mathcal{B} \ \ x \in B \ \ B \subset A$

Rem: $\mathcal{B} \subset \mathcal{A}$

Lemma $A \in \mathcal{A} \iff A = \bigcup \{ B \in \mathcal{B} : B \subset A \}$

$\Rightarrow A \supset \bigcup \qquad A \subset \bigcup$

$\Leftarrow A = \bigcup$

$\Downarrow$

$\forall x \in A \ \ \exists B \in \mathcal{B} \ \ B \subset A \ \ B \ni x$

$\Downarrow$

$A \in \mathcal{A}$.

Lemma $A \in \mathcal{A} \iff \exists \mathcal{B}' \subset \mathcal{B} \ \ A = \bigcup \mathcal{B}'$

$\Rightarrow \quad \mathcal{B}' = \{ B \in \mathcal{B} : B \subset A \}$

$\Leftarrow \quad A = \bigcup \mathcal{B}' \quad \forall x \in A \ \ \exists B \in \mathcal{B}' \ \ x \in B$

$B \subset \bigcup \mathcal{B}' = A$

$\Rightarrow A \in \mathcal{A}$. ∎

so open sets are union of basis elements

Lemma $\mathcal{A}(\mathcal{B})$ is a top.

Pf. $\mathcal{A}(\mathcal{B}) = \{ \bigcup \mathcal{B}' : \mathcal{B}' \subset \mathcal{B} \}$

$\mathcal{B}' = \emptyset \to \emptyset \in \mathcal{A}$

$\mathcal{B}' = \mathcal{B}$ use B1) $\Rightarrow X \in \mathcal{A}$

$\{A_\alpha\}_{\alpha \in I} \subset \mathcal{A} \qquad A_\alpha = \bigcup \mathcal{B}_\alpha \quad \mathcal{B}_\alpha \subset \mathcal{B}$

$$\bigcup_\alpha A_\alpha = \bigcup_\alpha \bigcup \mathcal{B}_\alpha = \bigcup \underbrace{\bigcup_\alpha \mathcal{B}_\alpha}_{\subset \mathcal{B}} \in \mathcal{A}.$$

assume $A', A'' \in \mathcal{A}$.

$A' = \bigcup \mathcal{B}' \qquad A'' = \bigcup \mathcal{B}'' \qquad \mathcal{B}', \mathcal{B}'' \subset \mathcal{B}$

let $x \in A' \cap A''$. $\to \exists B' \in \mathcal{B}'$, $B'' \in \mathcal{B}''$ $\quad x \in B' \cap B''$

$\qquad\qquad \mathcal{B}'' \subset \mathcal{B}$ $\qquad B' \subset A'$, $B'' \subset A''$

$\overset{B2)}{\Longrightarrow} \exists B''' \in \mathcal{B} \subset \mathcal{A} \qquad x \in B''' \subset B' \cap B''$

$x \in B''' \subset A' \cap A''$

$\forall x \in A' \cap A'' \quad \exists B''' \in \mathcal{A} \quad x \in B''' \subset A' \cap A''$

$\Rightarrow A' \cap A''$ is open $\Rightarrow A' \cap A'' \in \mathcal{A}$. ∎

<u>Lemma (B.2)</u> $\qquad (X, \mathcal{A}) \quad \mathcal{C} \subset \mathcal{A}$

how to identify a basis

$\forall O \subset \mathcal{A} \quad x \in O \quad \exists U \in \mathcal{C}$.

$x \in U \subset O$

$\Rightarrow \mathcal{C}$ is a basis

$\mathcal{B}$ $\quad\quad$ $\mathcal{B}'$

Lemma $(X, \mathcal{A})$, $(X, \mathcal{A}')$

$\mathcal{A}' \supset \mathcal{A} \iff \forall x \in X \ \forall B \in \mathcal{B}$

$\mathcal{A}'$ finer $\quad x \in B \implies \exists \ B' \in \mathcal{B}'$

$x \in B' \subset B$

Def: $(\mathbb{R}, <)$ $\quad \mathcal{B} = \{ (a,b) : a<b \}$

$\quad\quad\quad\quad\quad\quad\quad\quad \{x : a<x<b\}$

top generated by $\mathcal{B}$ is called standard top (Euclidean)

on $\mathbb{R}^1$ assumed below by default

Def: $\mathcal{B}' = \{ [a,b) : a<b \}$

$\mathbb{R}_\ell = (\mathbb{R}, \mathcal{B}')$ lower limit topology

Lemma $(X, \mathcal{A})$ Basis $\mathcal{B}$ $\quad Int(A) = \bigcup \{ B \in \mathcal{B} : B \subset A \}$

Ex. 1) $A = (0,1)$ $\quad A \subset \mathbb{R}$

$\bar{A} = [0,1]$ $\quad Int(A) = A = (0,1)$ (A open)

$Bd\, A = [0,1] \setminus (0,1) = \{0,1\}$

2) $A = \{1\}$ $\quad \bar{A} = A \quad Int(A) = \emptyset$

$Bd\, A = A$

3) $A = [0,1)$ $\quad \bar{A} = [0,1]$ $\quad Int(A) = (0,1)$

4) $A = \mathbb{Q}$ $\quad \exists a', a'' \in (a,b)$

$a' \in \mathbb{Q} \quad a'' \in \mathbb{R} \setminus \mathbb{Q}$

$\bar{A} = \mathbb{R} \quad Int(A) = \emptyset$

$Bd(A) = \mathbb{R}$ rat. & irrat #s are <u>dense</u>

<u>Ex.</u> $\mathbb{R}_\ell$

1) $A = [0,1]$ $\quad Int(A) = [0,1)$ A open!

$\bar{A} = A \quad Bd\, A = \{1\}$

2) $A = [0,1)$ $\quad Int\, A = A = [0,1)$ A open and closed!

$\bar{A} = A \quad Bd\, A = \emptyset$

3) $A = \{1\}$ $\quad Int\, A = \emptyset$

$\bar{A} = A \quad Bd\, A = \{1\}$

4) $A = \mathbb{Q}$ like prev. ex.

<u>Def.</u> X set $\quad \mathcal{S}$ subbasis $\quad \mathcal{S} \subset \mathcal{P}(X)$

top. of X is the top. with basis $\quad \bigcup \mathcal{S} = X$

$\mathcal{B} = \{ \bigcap \mathcal{S}' : \mathcal{S}' \subset \mathcal{S} \ |\mathcal{S}'| < \infty \}$

finite $\cap$s of els in $\mathcal{S}$

$\mathcal{A}$ = unions of <u>finite</u> $\cap$s of els in $\mathcal{S}$

§14 The Order Topology

1st major source of important topologies!

Def: $(X,<)$ ordered set $a<b$

$(a,b) = \{ x \mid a<x<b \}$
$[a,b)$ $\leq$ etc.

(had before)

Def: $(X,<)$ o.s. the order top. $\mathcal{A}$ on X is the one with basis $\mathcal{B}$ consisting of

1) all open intervals (a,b) $a<b$
2) $[a_0,b)$ a_0 smallest el. $b>a_0$ (if $\exists a_0$!)
$(a,b_0]$ b_0 largest el. $a<b_0$ (if $\exists b_0$!)

Ex. $|X|<\infty$ X ordered $\Rightarrow$ $\mathcal{A}$ discrete top.
similarly $(\mathbb{Z}_+,<)$

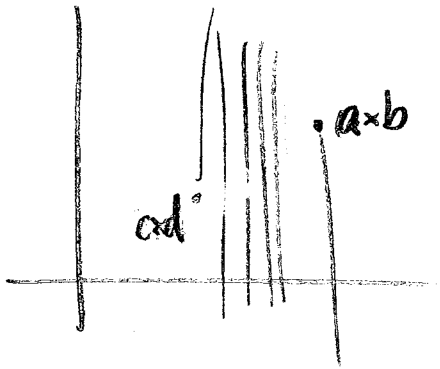
Ex. $(\mathbb{R},<)$ $\mathcal{A}$ = Euclidean top. on $\mathbb{R}$

Ex. $\mathbb{R}\times\mathbb{R}$ with dictionary order

recall: $a\times b > c\times d \Leftrightarrow a>c$ or $(a=c$ and $b>d)$

no smallest or largest el.

$$(c\times d, a\times b) = \begin{cases} \{c\}\times(d,b) & a=c \\ \{c\}\times(d,\infty) \cup (c,a)\times\mathbb{R} \cup \{a\}\times(-\infty,b) & c<a \end{cases}$$



Ex. $\{1,2\} \times \mathbb{Z}_+$ dict order

$1 \times n = a_n \qquad 2 \times n = b_n$

$X = a_1 \cdots a_n \cdots b_1 \, b_2 \cdots$

$(b_1, b_3) = \{b_2\}$ open $\qquad \{b_2\}$ open

$\{a_1\} = [a_1, a_2)$ open $\qquad \{a_1\}$ open

has sm. element

Similarly $\{a_i\}$ open $i > 1$

$\{b_j\}$ open $j > 2$

but $\{b_1\}$ is not open

if $b_1 \in B$ open $\exists a,b$: $b_1 \in (a,b) \subset B \quad b > b_1$

$a < b_1 \quad a = a_n$ for some n

$B \supset (a,b) \supset \{a_{n+1}, a_{n+2}, \cdots\}$

$|B| = \infty$

Def: $(X,<)$ $\quad a \in X$ define $(a,+\infty) = (a,\infty) = \{x : x > a\}$

$[a,\infty)$ $\qquad \geq$

$(-\infty, a) \qquad [-\infty, a]$ rays

(a,∞) $(-\infty,a)$ open rays (they are open sets)

$[a,\infty)$ $(-\infty,a]$ closed rays (closed sets)

§15 Product topology

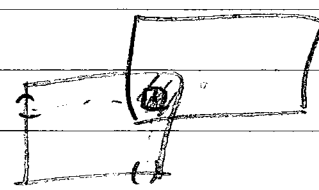
<u>Def:</u> $(X,\mathcal{A}), (Y,\mathcal{B})$ define a top.

$\mathcal{A}\times\mathcal{B}=\mathcal{C}$ on $X\times Y$ by the basis

$$\{A\times B : A\in\mathcal{A},\ B\in\mathcal{B}\}$$

$\mathcal{C}$ is called the product top. on $X\times Y$

<u>Th:</u> If $\mathcal{A}'\subset\mathcal{A}$ is a basis for $\mathcal{A}$
$\mathcal{B}'\subset\mathcal{B}$ is a basis for $\mathcal{B}$



$$\Rightarrow \{A'\times B' : A'\in\mathcal{A}',\ B'\in\mathcal{B}'\}$$

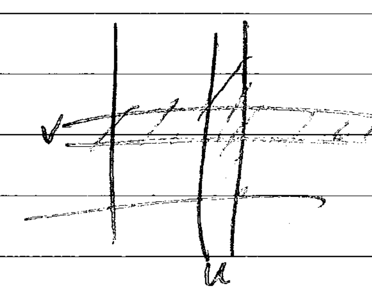
is a basis for $\mathcal{A}\times\mathcal{B}$.

<u>Def:</u> $\pi_1 : X\times Y \to X$ $\quad \pi_1(x\times y)=x$
$\pi_2 : X\times Y \to Y$ $\quad \pi_2(x\times y)=y$

projection

<u>Th:</u> $S = \{\underbrace{\pi_1^{-1}(U)}_{U\times Y} : U\subset X \text{ open}\} \cup \{\underbrace{\pi_2^{-1}(V)}_{=X\times V} : V\subset Y \text{ open}\}$

is a subbasis for $\mathcal{A}\times\mathcal{B}$



§16 The subspace top.

$(X, \mathcal{A})$ top. space $\quad Y \subset X$

my notation

$(Y, \mathcal{A}')$ with $\mathcal{A}' = \{A \cap Y : A \in \mathcal{A}\} =: \mathcal{A}_Y$

is called subspace top.

Lemma $\quad (X, \mathcal{A})$ basis $\mathcal{B} \quad Y \subset X$

$\{B \cap Y : B \in \mathcal{B}\}$ is a basis of $(Y, \mathcal{A}_Y)$

Def: If $X \supset Y \supset U$

we can say U open in Y if $U \in \mathcal{A}_Y$

closed $\qquad X \setminus U \in \mathcal{A}_Y$

Ex. $\quad X = \mathbb{R} \quad Y = (0,1] \quad U = (0,1]$

$U = Y$ is open in Y

Y is not open (in X)

U is not open in X

Lemma $\quad X \supset Y \supset U \quad U$ open in Y, Y open in X

$\Rightarrow U$ open in X

Th. $\quad A \subset X \quad B \subset Y \quad (X \times Y, \mathcal{C}) \quad \mathcal{C} = \mathcal{A} \times \mathcal{B}$

$\quad\;\; \mathcal{A} \qquad\;\; \mathcal{B} \qquad$ prod. top. $\quad \mathcal{A}_A \times \mathcal{B}_B = \mathcal{C}_{A \times B}$

Ex. $I = [0,1]$

$I_o^2 = I\times I$ with dict order top will be called

<u>the ordered square</u>

$I_o^2 = I\times I \subset \mathbb{R}^2$ but top. on I_o^2 is diff from

subspace top of $\mathbb{R}^2$

Ex. $A = \{1/2\}\times(0,1) = (1/2\times 0,\ 1/2\times 1)$

is open in I_o^2. Now cons top rel to

$\mathbb{R}^2$. Let $p = 1/2\times 1/2 \in A$

assume $\exists B \ni 1/2\times 1/2 \quad B\in \mathcal{B} = \{(a,b)\times(c,d) : \cap [0,1]^2\}$

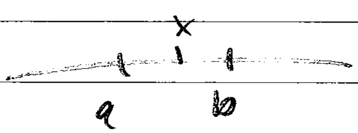
$\Rightarrow a<1/2,\ b>1/2 \Rightarrow B\not\subset A \quad \Rightarrow p\notin \mathrm{Int}(A)$ basis in std top.

In fact similarly you see $\mathrm{Int}(A)=\emptyset$

Def: $(X,<)$ $Y\subset X$ convex

$\Longleftrightarrow \forall a,b\in Y \quad a<b$

$\forall x\in X \quad a<x<b \Rightarrow x\in Y$



(distinguish from $A\subset V$ convex V VS/$\mathbb{R}$)

Rem: intls and rays are convex

but not the other way around.

Ex. $X = \mathbb{R} \setminus \{0\}$

$Y = (-\infty, 0)$

this is convex subset in X

but one cannot write $Y = (-\infty, a)$

$(-\infty, b]$

(a, b) ...

for a, b in X (not in $\mathbb{R}$)

$$\begin{tikzcd}
(X, <) \arrow[d] & Y \subset X \arrow[r, squiggly, "\text{restr. order}"'] & (Y, <|_{Y \times Y}) \arrow[d] \\
(X, \mathcal{O}_<) \arrow[rr, squiggly, "\text{restricted top.}"'] & & (Y, \mathcal{O}_{<'})
\end{tikzcd}$$

$<' = <|_{Y \times Y}$

order top.

Theo $\mathcal{O}_{<|_{Y \times Y}} = (\mathcal{O}_<)_Y$.

Convention $(X, <)$ $Y \subset X$ assumed with subspace top. $(\mathcal{O}_<)_Y$

$(= \mathcal{O}_{<|_{Y \times Y}}$ if Y convex! $)$ see 70

Ex. $\mathbb{Q} \subset \mathbb{R}$ subspace top. from $(\mathbb{R}, <)$ has a basis $\mathcal{B}_1$ $\{(a, b) \cap \mathbb{Q} \quad a, b \in \mathbb{R}\}$

order top. from $(\mathbb{Q}, <)$ has a basis these bases are not the same (e.g. $\mathbb{Q} \cap (0, \pi)$)

if however $\mathcal{B}_2 \subset \mathcal{B}_1$

and you can show all sets in $\mathcal{B}_1$ are unions of sets in $\mathcal{B}_2$ so the topologies of $\mathcal{B}_1, \mathcal{B}_2$ are the same

but this shows for general ordered sets be careful

§17 Closed sets and ~~limit~~ points
accumulation pts.

already defined closed sets

Th. $\emptyset, X$ closed

~~A, B closed $A \cap B$ closed~~

$\{A_i\}_{i \in I}$ A_i closed $\Rightarrow \bigcap_i A_i$ closed

Th. $X \supset Y \supset A$ A closed in $Y \Leftrightarrow \exists A' \subset X$ closed $A = A' \cap Y$.

Th. $X \supset Y \supset Z$ Y closed in X, Z closed in Y $\Rightarrow Z$ closed in X

$A \subset X$

$\bar{A} = \bar{A}^{*} = \{ x \in X \ \forall O \in \mathcal{A} : O \ni x \Rightarrow O \cap A \neq \emptyset \}$

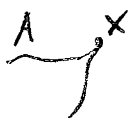
! $=$ $O \in \mathcal{B}$

$\mathcal{B}$ basis of $\mathcal{A}$

Th. $Y \subset X$ $A \subset Y$ $(Y, \mathcal{A}_Y)$ rel. top.

$$\overline{A}^{\mathcal{A}_Y} = \overline{A}^{\mathcal{A}} \cap Y$$

Def. $A \subset X$ $x \in X$ if $x \in \overline{A \setminus \{x\}}$



$\left(\iff \forall O \in \mathcal{A} \quad O \ni x \Rightarrow O \cap (A \setminus \{x\}) \neq \emptyset \right)$

call x a accumulation point of A
적집점

$A_{acc} = \{$acc. pts of $A\}$

if $x \in A$ and x is not an accum. point of A
call x an isolated point of A
고립점

A is discrete(이산집합) if all its pts are isolated

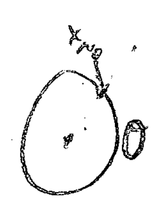
Th. $\overline{A} = A \cup A_{acc}$

Th. X is T_1 $x \in X$ is acc. pt of $A \iff$ for every neighborhood $O \ni x$ $|O \cap A| = \infty$

Convergence care is needed with limit and convergence

Def: $X \ni x_1, x_2, \dots, x_n, \dots$

we say $\lim_{n \to \infty} x_n = x$ $x_n \to x$



if $\forall O \ni x$ $O \in \mathcal{A}$ $\exists N_0$ $\forall n \geq N_0$ $x_n \in O$

Ex. $x_n = n$ in $\mathbb{R}$ with f.c. top. $\mathcal{O}$

let $a \in \mathbb{R}$ $\quad \mathcal{O} \ni a \quad \mathcal{O} \neq \emptyset$

$\Rightarrow |\mathbb{R} \setminus \mathcal{O}| < \infty \quad \overset{|\{x_n\}| = \infty}{\exists N} \quad \forall n \geq N \quad x_n \in \mathcal{O}$

$x_n \to a$ so $x_n = n \to a \quad \forall a \in \mathbb{R}$

Th. X is T_2 (Hausdorff)

$\Rightarrow$ limit (if it exists!) is unique

i.e. every seq of pts converges to at most one pt. (Unique limit Prop.)

Rem: converse is false; see ex (17a - 17b)

Th. X ordered $\Rightarrow X$ is T_2 in order top.

$(X, \mathcal{A})$ is $T_2 \quad Y \subset X \Rightarrow (Y, \mathcal{A}_Y)$ is T_2

$(X, \mathcal{A}) \; (Y, \mathcal{B}) \; T_2 \Rightarrow (X \times Y, \mathcal{A} \times \mathcal{B})$ is T_2

!! If $\exists x_1, x_2, \ldots \quad x_n \in A \quad x_n \to x \quad x_n \neq x$

then x is an accumulation pt of A

but the converse is false $\qquad A_{lim} \subset A_{acc}$

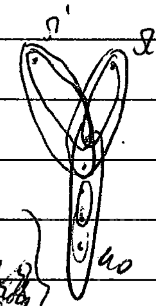
Def: limit pts $A_{lim} := \{ x \in X : \exists x_n \in A \setminus \{x\} \; x_n \to x \}$ = I DO NOT USE LIKE IN BOOK

$A_{lim} \neq A_{acc}$ can happen (I will give you ex later)

this is why be careful you understand well how a "limit point" is meant! (return to this p. 85-87)

In "metrizable" spaces like $\mathbb{R}, \mathbb{R}^n$ with Euclidean top. it's ok

$X = \underbrace{S_\Omega \cup \{\Omega\}}_{\bar{S}_\Omega} \cup \{\Omega'\}$



$\mathcal{A}$ with top basis $\left\{ \begin{array}{l} (a_0, b) \\ (a, \Omega] \\ (a, b) \end{array} \right.$ $\begin{array}{l} a, b \in \Omega \\ a < b \end{array}$ $\Big\} \cup \{ (a, \Omega) \cup \{\Omega'\} \ a \in S_\Omega \}$

basis of order top on $\bar{S}_\Omega$ "duplicate" Ω

$\mathcal{A}_{\bar{S}_\Omega}$ = order top on $\bar{S}_\Omega$ $(\Rightarrow T_2)$

$\mathcal{A}_{X \setminus \{\Omega\}}$ = ord. top on $\bar{S}_\Omega$ with $\Omega \to \Omega'$

X is not T_2 bec. any two basis elts $\ni \Omega, \ni \Omega'$ intersect

but X has the unique limit property

let $(x_n) \subset X$.

~~Lemma $x_n \to \Omega$ (resp. Ω') $\Leftrightarrow \exists N$ $\forall n \geq N$ $x_n = \Omega$ (resp. Ω')~~

~~Pf. Assume $\exists$ inf many $x_n \in S_\Omega \cup \{\Omega'\}$~~

~~If inf many $x_n = \Omega' \Rightarrow \exists x_{n_k} = \Omega'$, $\Omega' \notin U \ni \Omega$ $\exists$ open~~

~~$\Rightarrow x_n \not\to \Omega$ ↯~~

~~So $\exists$ inf many $x_n \in S_\Omega$~~ $\{x_{n_k}\} \subset S_\Omega$ countable

prop of S_Ω $\Rightarrow$ bounded. $x_{n_k} \leq d$ $d \in S_\Omega$ $x_{n_k} \notin (d, \Omega]$

$x_{n_k} \not\to \Omega \Rightarrow x_n \not\to \Omega$ ↯ $\square$

Now assume $(x_n) \subset X$ ~~$x_n \to x^1, x^2 \in X$ $x^1 \neq x^2$~~

If $x^1 \in S_\Omega \Rightarrow (x_n)$ bounded $x^2 \neq \Omega, \Omega'$

$x_n \to x^1, x^2$ in $S_\Omega \in T_2$ space $\Rightarrow x^1 = x^2$

Similarly if $x_2 \in \Omega$.

(45)

So remains to test

$\{x_2^1, x_2^2\} = \{\Omega, \Omega'\}$

but this given by key pole lemma.

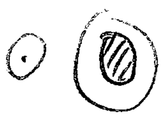
Thus $X_1 = X_2$ and X has unique lim. property

Th. $(x_n, y_n) \to (x,y) \iff$ in X $x_n \to x$ in ...
and in Y $y_n \to y$ in $\mathcal{B}$

separation axioms (§31)

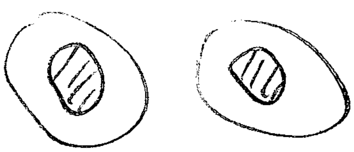
Def: A space $(X, \mathcal{A})$ is regular if $\quad$ p.194

$x_1 \quad A = \bar{A} \not\ni x \quad \exists\, O_1 \ni x \quad O_i \in \mathcal{A}$
$\quad O_2 \supset A \quad O_1 \cap O_2 = \emptyset$



Def: A space $(X,\mathcal{A})$ is normal if

$\forall A_1, A_2 \quad A_1 = \bar{A_1} \quad A_2 = \bar{A_2} \quad A_1 \cap A_2 = \emptyset$
$\quad \exists\, O_1, O_2 \in \mathcal{A} \quad O_1 \cap O_2 = \emptyset \quad O_i \supset A_i$



If pts are closed then normal $\Rightarrow$ regular $\Rightarrow T_2$
$\Updownarrow$
T_1 $\quad$ but not always otherwise!

So we have a diagram

$T_2 \Rightarrow T_1 \Rightarrow T_0$

regular $\Leftarrow$ normal

! In other books
T_3 = normal
T_4 = regular

Def: T_3 is regular Hausdorff (= regular Fréchet)
T_4 is normal Hausdorff (= normal Fréchet)

Th. order topologies are normal and T_2 (and hence,

(1) Convergence behavior

X space

A top. on X $\longmapsto$ convergence behavior of A $\mathcal{C}(A)$

$A \subset \mathcal{P}(X)$ $\quad (x_i) \in X^\omega \longmapsto \{\text{limits of } (x_i)\}$

$= \{x \in X : x_i \to x\}$

$A \in \{\text{top. on } X\} \subset \mathcal{P}(\mathcal{P}(X))$ $\quad \in \mathcal{P}(X)$

$\quad\quad \mathcal{T}(X)$

conv. behavior functor $\quad \mathcal{T}(X) \xrightarrow[\mathcal{C}]{} \mathcal{F}(X^\omega, \mathcal{P}(X))$

Qu: Does the conv. beh. determine the top.

i.e. is the conv. beh. functor injective?

Ex. (X^ω, A) $\quad$ you give $f_n \to f$ in $A \Leftrightarrow$

$f_n \to f$ pointwise

$\Rightarrow A =$ prod top?

o X metrizable $\quad$ ... $f_n \rightrightarrows f$ uniform

$\Rightarrow A =$ uniform top?

The answer is no! in general

i.e. one cannot identify a top. from its

conv. behavior alone!

Ex. Con $X = \overline{S_\Omega} \cup \{\Omega\} = S_\Omega \cup \{\Omega, \Omega'\}$. Define a top.

A on X with top. basis $\quad$ let $\tilde{\mathcal{B}} = \{ (a, b) \quad a < b \quad a, b \in \overline{S_\Omega} \}$

thus

$\cup \{ [a, \ldots) \quad b \in \overline{S_\Omega} \} \cdot \{ (a, \Omega'] : a \in \ldots \}$

$\mathcal{B} = \tilde{\mathcal{B}} \cup \{(B \setminus \{\Omega\}) \cup \{\bar{\Omega}\} : \Omega \in B\}$.

This is the top. as in the previous example.

$\mathcal{A} = \mathcal{A}(\mathcal{B})$ is not Hausdorff but has unique limit property

Now cons $\mathcal{A}' = \mathcal{A}(\mathcal{B}')$ with $\mathcal{B}' = \mathcal{B} \cup \{\{\Omega\}\}$

(again to check easily $\mathcal{B}'$ is a base)

$\mathcal{A}'$ is Hausdorff but $\mathcal{A}$ and $\mathcal{A}'$ have same convergence behavior : $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}')$

$x_n \to x$ in $\mathcal{A} \Leftrightarrow x_n \to x$ in $\mathcal{A}'$

Thus the convergence behavior does not det. the top., not even whether it is T_2

$\Rightarrow$ not even up to self-~~transform~~ fixing conv. behavior

[$h : X \to X$ bijective

$\mathcal{A} \mapsto \{h(A) : A \in \mathcal{A}\} = h(\mathcal{A})$ is a top.

$$\begin{array}{cc} \mathcal{F}(X^{\mathbb{N}}, \mathcal{P}(X)) & \mathcal{C}(\mathcal{A}) \mapsto h(\mathcal{C}(\mathcal{A})) \\ \circlearrowleft \quad \circlearrowleft & \parallel \\ h \quad\ \ h & \mathcal{C}(h(\mathcal{A})) \end{array}$$

If $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}')$ $\exists h : X \to X$ bij

$\mathcal{A}' = h(\mathcal{A})$ s.t. $\mathcal{C}(\mathcal{A}) = \mathcal{C}(h(\mathcal{A}))$?

even this answer is <u>no</u>!]

Remark: You can use as well

$X = \bar{S}_\Omega \quad \mathcal{A} = \mathcal{A}(\tilde{\mathfrak{B}})$

$\mathcal{A}' = \mathcal{A}(\tilde{\mathfrak{B}} \cup \{\{\Omega\}\})$

Then $\mathcal{A} \neq \mathcal{A}'$ but $\ell(\mathcal{A}) = \ell(\mathcal{A}')$

however in this case obv. both $\mathcal{A}, \mathcal{A}'$ are T_2 and I don't know if $\nexists$ self-transformation h on $\bar{S}_\Omega$ with $h(\mathcal{A}) = \mathcal{A}'$. Likely not, but it requires some arg. to prove, while in our way we get this + T_2-independence readily

for ex. $\mathcal{A} \ni \{x\}$ one el. set

$\iff$ x has no immed. predecessor or $x = a_0$

(x has also an immed. succ. as $x \neq \Omega$)

but $\mathcal{A}$ contains uncountably many 1-el. sets (old exercise)

so then one could argue with # of 1-el. sets that $\nexists h$ etc ...

(2) Limit points

Exercise Let X be a top space

$x_n \in \mathcal{F}(\mathbb{Z}_+, X)$ a seq.

Let $h: \mathbb{Z}_+ \to \mathbb{Z}_+$ bijection

then $x_n \to x$ in X $\iff$ $x_{h(n)} \to x$ in X

This means ordering a seq is not relevant for its limit(s)

Thus the limit points of a seq

$\{x \in X : x_n \to x\}$

can in fact be defined on the set $\{x_n\}$

Def. X top space $A \subset X$ $|A| = \omega$

define $A^{lim} = \{ x \in X : \exists h: \mathbb{Z}_+ \to A$ bijection

$h(n) \xrightarrow[n\to\infty]{} x \}$

Note, however, that $A^{lim} \neq A_{lim}$

the set of limit point of A as a set, as defined in the closure section

In fact $A_{lim} = \bigcup_{A' \subset A, |A'| = \omega} (A')^{lim}$, (if X is T_1)

also for any subset $A \subset X$ (not necessarily on cardinality ω)

§20 The Metric Topology

Most important and fund source of topologies

Def: X set $\quad d: X \times X \to [0,\infty)$

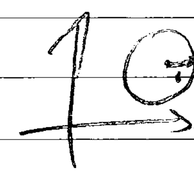
distance fn

$d(x,x)=0 \qquad d(x,y)>0 \quad x\neq y$

$d(x,y)=d(y,x)$

$d(x,y)+d(y,z) \geq d(x,z)$

$d(x,y)$ distance betw x and y

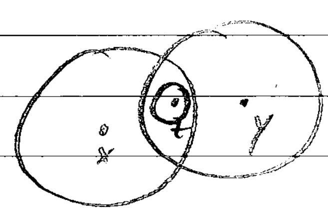


$\varepsilon>0$, $x\in X$ $\qquad B_\varepsilon(x) = \{ y\in X \ \ d(x,y)<\varepsilon \}$

(open) ball $\quad$ ε-ball centered at x

Def: The metric top. $\mathcal{T}_d$ on (X,d) "induced by d" is the one with basis $\{ B_\varepsilon(x) : \ x\in X \ \ \varepsilon>0 \}$

to check: balls form a basis



B2) $z\in B_\varepsilon(x)\cap B_{\varepsilon'}(y)$

$B_{\varepsilon''}(z) \subset B_\varepsilon(x)\cap B_{\varepsilon'}(y)$

for $\varepsilon'' = \min(\varepsilon - d(x,z), \ \varepsilon' - d(y,z)) > 0$

Def: $(X,\mathcal{T})$ metrizable $\iff \exists\, d$ on X ...

Ex. X, $\delta(x,y) = \begin{cases} 0 & x=y \\ 1 & x\neq y \end{cases}$

discrete distance $B_1(x) = \{x\}$ so

it induces the discrete top $\mathcal{A} = \mathcal{P}(X)$

explains the name

$A \subset X$ every $a \in A$ is discrete point

every set is discrete

so the discrete top is metrizable

Ex. standard metric on $\mathbb{R}$ $d(x,y) = |x-y|$

induces the usual (Euclidean) top.

bec. $(a,b) = B_\varepsilon(x)$ for $x = \frac{a+b}{2}$ $\varepsilon = \frac{b-a}{2}$

Euclidean top. is metrizable

$\Rightarrow$ non-uniqueness $\hat{d}(x,y) = 2|x-y|$ induces the same top.

Ex. $\mathbb{R}$ f.c. top not m'ble

if were $\bigcap_{n=1}^{\infty} B_{1/n}(x) = \{x\}$ (open)

$\underbrace{\mathbb{R}\setminus\{x\}}_{\text{uncount.}} = \underbrace{\bigcup_{n=1}^{\infty}(\underbrace{\mathbb{R}\setminus B_{1/n}(x)}_{\text{finite}})}_{\text{count.}}$

Def: V VS over $\mathbb{R}$ or $\mathbb{C}$

a norm $\|\cdot\| : V \to [0,\infty)$

satisfies $\|v\| = 0 \iff v = 0$

$\|\lambda v\| = |\lambda| \cdot \|v\|$ $\lambda \in \mathbb{R}(\mathbb{C})$ $v \in V$

$\|v\| + \|w\| \geq \|v+w\|$

E.g. $\|v\| = \sqrt{\langle v,v\rangle}$ if $\exists \langle , \rangle$

$\|\cdot\|$ induces a metric by $d(x,y) = \|x-y\|$

{inner prod sp} ⊂ {normed sp} ⊂ {metric sp} ⊂ {top. sp}

all ⊊ we see later

Ex. $\mathbb{R}^n$ with Euclidean norm

$\|\vec{x}\| = \|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$

induces the product top. on $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\dots\times\mathbb{R}$

basis are $\{(a_1,b_1)\times\dots\times(a_n,b_n) \quad a_i<b_i\}$

Ex. the Hölder p-norms

$$\|\vec{x}\|_p = \sqrt[p]{|x_1|^p+\dots+|x_n|^p} \quad \text{for } p\in[1,\infty)$$

are not mult. eq. of $\|.\| = \|.\|_2$ for $p\neq 2$

but all induce the same topology on $\mathbb{R}^n$

Def: (X,d) metric space $A\subset X$ $x\in X$ $\operatorname{dist}(x,A) = \inf\{d(x,a) \ a\in A\}$

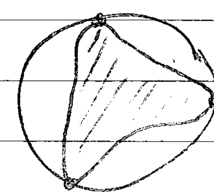
lemma: $\operatorname{dist}(x,A)=0 \iff x\in\bar{A}$

Def: (X,d) metric space $A\subset X$

$$\operatorname{diam} A = \sup\{d(a,a'),\ a,a'\in A\}$$

Ex. $\operatorname{diam} B_\varepsilon(x) \leq 2\varepsilon$ (not alw "=" !)

diam can be ∞ if d is unbounded



e.g. $\operatorname{diam}(\mathbb{Z}_+) = \infty$ $\mathbb{Z}_+\subset\mathbb{R}$ with Eucl. metric

sometimes it is useful to bound d:

Research pb.
$A\subset\mathbb{R}^2$ $\operatorname{diam}(A)\leq 1$
$A\subset \bar{B}_{1/\sqrt{3}}(x)$ for some x?

$1/\sqrt{3}$ smallest poss.

Thm let (X,d) metric space

Define $\bar{d}: X\times X\to\mathbb{R}$ by

$$\bar{d} = \max(d,1)$$

standard bounded metric.

$\bar{d}$ induces $\mathcal{T}_{\bar{d}} = \mathcal{T}_d$, same top.

Lemma (X,d), (X,d') $\mathcal{A}_{d'} \supset \mathcal{A}_d$

$\iff \forall x \in X \ \varepsilon > 0 \ \exists \delta$

$B_{d',\delta}(x) \subset B_{d,\varepsilon}(x)$

Lemma $A \subset (X,d)$ $\mathcal{A}_{d/A\times A}$ is the rel. top. of $\mathcal{A}_d$ of A. if $\mathcal{A}$ is metrizable $\Rightarrow \mathcal{A}_A$ is metrizable

Th. (X,d) top. space

$x_n \to x \iff \forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ d(x_n, x) < \varepsilon$

$x_n \in B_\varepsilon(x)$

Th. metric spaces are Hausdorff and normal (so anything else on chart p. (58)) T_4

Pf. modify the idea for order topologies (exercise)

Th. (X,d) metric $x \in \bar{A} \iff \exists x_n \in A \ x_n \to x$

Sequence lemma
i.e. trouble (ex (57)) does not occur for metrizable spaces

§18 continuous maps

$(X,\mathcal{A})$ $(Y,\mathcal{B})$

Def: $f : X \to Y$ top. spaces continuous

iff $\forall O \in \mathcal{B}$ open in Y $f^{-1}(O) \in \mathcal{A}$ open in X

continuous "relative to the top's $\mathcal{A}$ & $\mathcal{B}$."

Ex. $(X,\mathcal{A})$ $(X,\mathcal{B})$

$\mathcal{A}$ finer that $\mathcal{B} \iff id_X : (X,\mathcal{A}) \to (X,\mathcal{B})$

is continuous

Rem (p. 90) if $\forall O \in \mathcal{A}$ $f(O) \in \mathcal{B}$ call f open

let $(X,\mathcal{A})$ have basis $\mathcal{A}'$ (83)
$(Y,\mathcal{B})$ have basis $\mathcal{B}'$

Lemma f continuous $\iff \forall B' \in \mathcal{B}' \quad f^{-1}(B') \in \mathcal{A}$

$\iff \forall B' \in \mathcal{B}' \quad \forall x \in f^{-1}(B')$

$\exists A' \in \mathcal{A}' \quad x \in A' \subset f^{-1}(B')$

("ε-δ" & sequence condition)

Lemma If (X,d) (Y,d') are metric spaces can take balls as basis

$f: X \to Y$ continuous $\iff \forall \varepsilon > 0 \quad \exists \delta > 0$

$B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$

$\iff \forall x_n \to x$ in X $f(x_n) \to f(x)$ in Y

Th. X, Y top. spaces $f: X \to Y$. The foll are equivalent

1) f continuous
2) $\forall A \subset X \quad f(\bar{A}) \subset \overline{f(A)}$
3) $\forall B \subset Y$ closed $f^{-1}(B) \subset X$ closed
4) $\forall x \in X \quad \forall U \ni f(x)$ open $\exists V \ni x$ open

$V \subset f^{-1}(U) \quad (\iff f(V) \subset U)$

$(X,\mathcal{A}) \quad (Y,\mathcal{B})$

Def: f homeomorphism $f: X \to Y$ bij, f, f^{-1} continuous

$\mathcal{A} = \{ f^{-1}(B),\ B \in \mathcal{B} \}$

$\mathcal{B}$ dets. $\mathcal{A}$

$X \cong Y$ homeomorphic
if $\exists f: X \to Y$ homeomorphism

<u>Def</u>: $f: X \hookrightarrow Y$ cont inj
if $f^{-1}: f(X) \to X$ continuous, call
f a (top.) embedding

<u>Ex.</u> $f: [0,2\pi) \longrightarrow S^1 = \{ (x,y) : x^2+y^2=1 \}$
$f(t) = (\cos t, \sin t)$ with rel. top to $\mathbb{R}^2$


is bijective and cont's
but its inverse is not cont $\Leftarrow$ f is not open

$[0,\alpha)$ $T = f([0,\alpha)) \subset S^1$ is not open, $S^1 \cap B_\varepsilon(1) \not\subset T$
$\wedge$ $1 \in S^1$ but $\forall \varepsilon > 0$
$[0,2\pi)$ open

<u>Constructing Cont's Functions</u>

see p. 105, 106 in the book

<u>Pasting Lemma</u>

$X = A \cup B$ A, B closed in X
$f: X \to Y$ $f|_A : A \to Y$
$f|_B : B \to Y$ cont. $\Rightarrow f: X \to Y$ cont's