Maps into products

Th $f : A \to X \times Y$ (with product top) is continuous $\Leftrightarrow$

coordinate fcts. $f_1 : A \to X$ $\quad f_1 = pr_X \circ f$
$\quad\quad f_2 : A \to Y$ $\quad f_2 = pr_Y \circ f$

are continuous.

§19 Product topology II (1st part §15)

Motivation: $X_1 \times \dots \times X_n$ with top $\mathcal{A}_1 \dots \mathcal{A}_n$

defined product top by (1) subbasis (for $n=2$ but finite $n$ is same thing)

$$\bigcup_{k=1}^{n} \{ pr_k^{-1}(U) : U \subset X_k \text{ open} \} \text{ or}$$
$$pr_k^{-1}(U) = X_1 \times \dots \times X_{k-1} \times U \times X_{k+1} \times \dots \times X_n$$

(2) basis $\{ U_1 \times \dots \times U_n : U_i \subset X_i \text{ open} \}$

gives (same) product top

But what if $\infty$ product? There's a difference!

First let's define general tuples indexed by an arb. set.

Convention write $\mathcal{F}(X, Y)$ for $\{f : X \to Y\}$

Recall $\mathcal{F}(\mathbb{Z}_+, X) = \{ f : \mathbb{Z}_+ \to X \} \cong X^\omega = \{ (a_1, a_2, \dots) \; a_i \in X \}$
$\quad f(n) = a_n$

$$= \prod_{i=1}^{\infty} X$$

$$\prod_{i=1}^{n} X_i = \{ f : \mathbb{Z}_+ \to \bigcup X_i \; : \; f(n) \in X_n \; \forall n \}$$

For $J$ index set, $X$ set, define $J$-tuple of els in $X$ to be a function $x: J \to X$

$$x = (x_\alpha)_{\alpha \in J} \qquad x(\alpha) = x_\alpha$$

$$X^J = \{x : \} \simeq \{f : J \to X\}$$

$$\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$$

Def: $\prod_{\alpha \in J} A_\alpha = \{ f : J \to \bigcup A_\alpha : f(\alpha) \in A_\alpha \ \forall \alpha \in J \}$

$J$-indexed product

Def: $\{A_\alpha\}_{\alpha \in J}$ $(A_\alpha, \mathcal{T}_\alpha)$ top. spaces

the box topology on $A = \prod_{\alpha \in J} A_\alpha$ is defined by the basis

$$\left\{ U = \prod_{\alpha \in J} U_\alpha \quad \begin{matrix} U_\alpha \in \mathcal{T}_\alpha \\ \forall \alpha \in J \end{matrix} \right\}$$

"$U$ is a box" gen. of (2) prev. page

Def: $(A_\alpha, \mathcal{T}_\alpha)$ top spaces

the product top on $A = \prod A_\alpha$ is defined by the subbasis

$$\bigcup_{\alpha \in J} \bigcup_{U_\alpha \in \mathcal{T}_\alpha} pr_\alpha^{-1}(U_\alpha)$$

with $pr_\alpha : A \to A_\alpha$ being the proj on the

$\alpha$-th coordinate $\; pr((a_\beta)_{\beta\in J}) = a_\alpha$ general (1) (18)

The product topology will be assumed by default!

Obs. box top $\supset$ product top.

but box top is quite strong and often good for cex while product top is used in many theorems.

For metric spaces $(A_\alpha, d_\alpha)$ there is

one more important top on $\prod A_\alpha$, the

uniform top. It has basis being boxes of "equal length"

$$\mathcal{B} = \left\{ \prod_{\alpha\in J} B_\varepsilon(x_\alpha) \quad x_\alpha \in A_\alpha,\ \varepsilon > 0 \right\}$$

$\varepsilon$ does not depend on $\alpha$!

Th: the uniform top is induced by the uniform metric

bounded metric

$$d\left( (a_\alpha)_{\alpha\in J}, (b_\alpha)_{\alpha\in J} \right) = \sup \left\{ \bar{d}_\alpha(a_\alpha, b_\alpha) : \alpha \in J \right\}$$

Convergence $\qquad$ Notation with $\mathcal{F}(X,Y)_{prod}$, $\mathcal{F}(X,Y)_{box}$, $\mathcal{F}(X,Y)_{uni}$

Th/ convergence in the product top is pointwise

Def. conv

$$f_n \to f :\Longleftrightarrow \forall \alpha\in J \quad f_n(\alpha) \to f(\alpha) \text{ in } A_\alpha$$

$\forall \alpha \in J \quad \forall O \in \mathcal{T}_\alpha \; O \ni f(\alpha) \quad \exists N \quad \forall n \geq N \quad f_n(\alpha) \in O$

Th. / Def: convergence in the uniform top is the uniform convergence

$f_n \rightrightarrows f :\Leftrightarrow \forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ \forall \alpha \in J$
$d_\alpha(f_n(\alpha), f(\alpha)) < \varepsilon$

Rem:

$f \to f \quad \forall \alpha \in J \ \forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ d_\alpha(f_n(\alpha), f(\alpha)) < \varepsilon$

What is convergence in the box topology?

Ex. consider $\{f : \mathbb{R} \to \mathbb{R}\}$ with box. top.

when $f_n \to 0$?

Ass $f_{n_1}(x_1) \neq 0$ choose $U_{x_1} = (-f_{n_1}(x_1), f_{n_1}(x_1))$
$f_{n_1} \notin U = \prod_{x \in \mathbb{R}} U_x$

Ass $\exists x_2 \neq x_1 \ n_2 > n_1 \ f_{n_2}(x_2) \neq 0 \qquad U_{x_2} = (-f_{n_2}(x_2), f_{n_2}(x_2))$
$f_{n_2} \notin U = \prod_{x \in \mathbb{R}} U_x$

$\leadsto$ find $(f_{n_1}, f_{n_2}, \dots) \notin U \ni 0$ open $\to f_n \not\to 0$
etc. when does this fail?

$f_n \to 0 \iff \exists N \quad S_N := \bigcup_{n \geq N} \{x : f_n(x) \neq 0\}$ finite

and $\quad \forall x \in S_N \quad f_n(x) \to 0$

$(\iff f_n|_{S_N} \to 0|_{S_N} \iff f_n|_{S_N} \rightrightarrows 0|_{S_N})$

Ex. $f_n(x) = \begin{cases} 0 & x \neq n \\ \frac{1}{n} & x = n \end{cases}$ (so box conv $\Rightarrow$ pointwise conv. and uniform) of course $f_n \to 0$ and even $f_n \rightrightarrows 0$

also $|\{x : f_n(x) \neq 0\}| = 1 < \infty \quad \forall n$

but $\bigcup_{n \geq N} \{x : f_n(x) \neq 0\} = \mathbb{Z}_+ \cap [N, \infty)$
infinite

so $f_n \not\to 0$ (and not to any other limit) in box top.

The box top is easily seen to be Hausdorff and I can prove regular. It is not known that normal
see Ex 5 p. 203

of $\mathcal{F}(J, X)$

Th. In product or box top (and hence in the uniform top as well, in $X_\alpha$ metric)

for $A_\alpha \subset X_\alpha$

$$\overline{\prod_{\alpha \in J} A_\alpha} = \prod_{\alpha \in J} \overline{A_\alpha}$$

Ex. Cons $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ in box top. $\mathcal{F} = \mathcal{F}(\mathbb{R}, \mathbb{R})_{box}$

let $A = \prod_{\alpha \in \mathbb{R}} (0,1) = \{f : \mathbb{R} \to \mathbb{R} \ \text{image}(\mathbb{R}) \subset (0,1)\}$

Then $\overline{A} = \prod_{\alpha \in \mathbb{R}} [0,1] = \{f : \mathbb{R} \to \mathbb{R} \ \text{image}(\mathbb{R}) \subset [0,1]\}$

but cons $A_{lim} = \{f \in \mathcal{F} : \exists f_n \in A : f_n \to f\}$

$$A_{lim} = \bigcup_{\substack{S \subset \mathbb{R} \\ |S| < \infty}} \prod_{\alpha \in \mathbb{R}} \begin{cases} [0,1) & \alpha \notin S \\ [0,1] & \alpha \in S \end{cases}$$

$$= \{ f : \mathbb{R} \to \mathbb{R} \quad \text{image}(f) \subset [0,1] \quad \exists S \subset \mathbb{R}\ |S| < \infty \quad \text{image}(f|_{\mathbb{R} \setminus S}) \subset [0,1) \}$$

E.g. $c \in A_{acc}$ $0 \notin A_{lim}$

$A_{lim} \subsetneq A_{acc} \Rightarrow$ seq. lemma fail's $\Rightarrow$ box top not metrizable

If time $\to$ example from 86

$\Rightarrow$ ( $\mathcal{A}' \supset \mathcal{A}$ $\mathcal{A}$ metrizable $\not\Rightarrow$ $\mathcal{A}'$ metrizable)
(finer, coarser, unlike $T_i$ $i \leq 2$ 44 for ex.)

Order topologies II

Th. $(X, <)$ ordered set $\Rightarrow$ order top is normal

Ex. $\mathbb{R} \times \mathbb{R}$ with dict order is metrizable

$\simeq [0,1) \times [0,1)$ 　 hw Ex. 7.20.2 p. 124

ordered square $I_0^2 = [0,1] \times [0,1]$ is not, proof later! see 80

i.e. restricting the order top of $\mathbb{R} \times \mathbb{R}$ to $[0,1] \times [0,1]$ does not give the same top. as the top from restricting the dict order on $[0,1] \times [0,1]$ see 54

(My) Pf. Let $(X, <)$ be ordered.
$A, B \subset X$ closed disjoint fixed.

Idea. $\forall a \in A$ construct $(x_a, y_a) \ni a$ 　 $(x_a, y_a) \cap B = \emptyset$
$b \in B$ 　 $(\bar{x}_b, \bar{y}_b) \ni b$ 　 $(\bar{x}_b, \bar{y}_b) \cap A = \emptyset$.
$\forall a, b$ 　 $(x_a, y_a) \cap (\bar{x}_b, \bar{y}_b) = \emptyset$ 　 $(\cdot)$

Def. Rule $O_1 = \bigcup_{a \in A} (x_a, y_a)$ $O_2 = \bigcup_{b \in B} (\bar{x}_b, \bar{y}_b)$

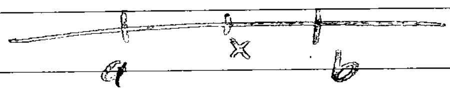
$O_1 \supset A$ $O_2 \supset B$ $O_1 \cap O_2 = \emptyset$.

Need to define $x_a, y_a, \bar{x}_b, \bar{y}_b$.

1st need to take care of blank intervals

Def: a) (recall) $I \subset X$ convex if

$\forall a, b \in I$ $a < b$ $\forall x \in X$ $a < x < b \Rightarrow x \in I$



b) $I \subset X$ is called blank interval (BI) if

$I$ convex and $I \cap (A \cup B) = \emptyset$

c) $I \subset X$ is maximal b.i. (MBI) if

$I$ is BI $\forall$ BI $I' \supset I \Rightarrow I' = I$

Lemma each BI $I \subset$ ! one MBI $\tilde{I} = M_I$

Pf. $\exists$ max BI $\hat{I} \supset I$ by max principle

unique bec if $\tilde{I}, \tilde{I}' \supset I$ $\hat{I} = \tilde{I} \cup \tilde{I}' \supset I$

MBI also interval

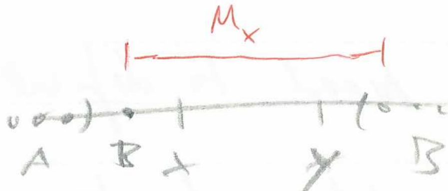
by max $\hat{I} \supset \tilde{I}$ $\tilde{I}$ max $\Rightarrow$ $\hat{I} = \tilde{I}$

$\hat{I} \supset \tilde{I}'$ $\tilde{I}'$ max $\Rightarrow$ $\hat{I} = \tilde{I}'$ $\Rightarrow \tilde{I} = \tilde{I}'$.

l.p. $\forall x \notin A \cup B$ $\{x\}$ is BI

$\exists!$ MBI $\supset \{x\}$ write it $M_x$

**Lemma:** $\forall x, y \in X$ $[x,y]$ BI $\iff M_x = M_y$




$\Leftarrow$

*Pf.* $M_x = M_y \supset \{x,y\}$ BI

$[x,y] \subset I_x \implies [x,y] \cap (A \cup B) = \emptyset$

$\implies [x,y]$ BI

$\implies$

$x \notin A \cup B$ let $I$ BI $I \ni x$

$I \subset$ MBI $M_I$

$x \in$ MBI $M_I$

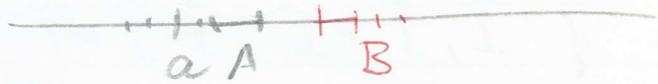
by uniqueness $M_I = M_x$

Now $x, y \in [x,y]$ BI by $\uparrow$ arg.

$M_{[x,y]} = M_x = M_y$ $\square$

**Conv.** For each MBI $\Delta \subset X$ fix an el. $\alpha_\Delta \in \Delta$. (AC!)

Return to def of $y_a$ (namely $M_{\alpha_\Delta} = \Delta$)




Assume $a \in A$. So $a \notin B = \overline{B}$

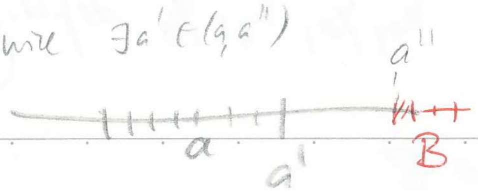
If $a = \max X$ set $y_a = \infty$ $(\ldots, \infty) = \ldots, a]$

$\exists a'' > a$ $a''$ is lower bound for $B \cap [a, \infty)$

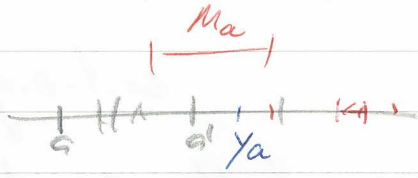
$B \cap (a, a'') = \emptyset$

[0] $a''$ is immed succ of $a$ set $y_a = a''$

otherwise $\exists a' \in (a, a'')$ (need to avoid $a' \in B$)




(1) If $a' \in A$ set $y_a = a'$




(2) If $a' \notin A \Rightarrow a' \notin A \cup B$ set $y_a = \alpha_{M_{a'}}$

Then $y_a$ has the foll properties

1) $y_a = \infty$ only if $a = \max X$

assume $a \neq \max X$ below

2) $y_a > a$ and $[a, y_a] \cap B = \emptyset$

(pf) If $y_a < a$ then need case [2] and

$$M_{y_a} = M_{\alpha_{M_{a'}}} = M_{a'}$$

$\Rightarrow [y_a, a']$ blank $a \in [y_a, a']$ $a \in A$ ↯

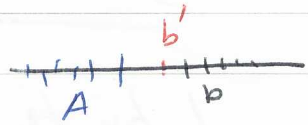
3) if $y_a \notin A$ then $y_a = \alpha_{M_{y_a}}$

Similarly $\overline{y}_b$ for $b \in B$

$\overline{x}_b$ should be defined so that

1) $\overline{x}_b = -\infty$ only if $b = \min X$

assume $b \neq \min X$




2) $\overline{x}_b < b$ and $[\overline{x}_b, b] \cap A = \emptyset$

3) if $\overline{x}_b \notin B$ then $\overline{x}_b = \alpha_{M_{\overline{x}_b}}$

Similarly $x_a$. So now $x_a, y_a, \bar{x}_b, \bar{y}_b$ defined. (74)

Now assume $a \in A$ $\quad b \in B$

w.l.o.g. $a < b$ $\quad \Rightarrow (y_a \neq \infty \quad \bar{x}_b \neq -\infty)$

(otherwise turn $A, B$ around in the below arg.)

assume $(x_a, y_a) \cap (\bar{x}_b, \bar{y}_b) \neq \emptyset$ to prove (*) by ↯ (70)

so $\bar{x}_b < y_a$

$[\bar{x}_b, b] \cap A = \emptyset$

$[a, y_a] \cap B = \emptyset$

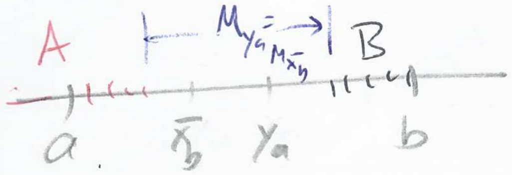
$[\bar{x}_b, y_a] \cap (A \cup B) = \emptyset$

is a BI

$M_{y_a} = M_{\bar{x}_b}$

$y_a \notin A \overset{31}{\Rightarrow} y_a = d_{M_{y_a}}$

$\bar{x}_b \notin B \overset{31}{\Rightarrow} \bar{x}_b = d_{M_{\bar{x}_b}}$




$y_a = \bar{x}_b$ ↯

$\Rightarrow (*) \Rightarrow$ ☐

## §26 Compact spaces

I try to discuss this 1st exclusively for metric spaces and say later how/what to generalize to top. spaces

Def: $x_n \subset (X,d)$ seq. $x_{n_m}$ is a subseq if $n_m > n_{m-1}$ $n_m \in \mathbb{Z}_+$

(75)

Def: (sequence) compactness / limit point compactness

$(X,d)$ is called (seq) compact if — see non-comp p. (78) !!

$\forall (x_n) \subset X$ seq. $\exists (x_{n_m}) \subset (x_n)$ subsequence

converging in $X$ $\exists x \in X$: $x_{n_m} \to x$ limit pt.

Rem $(X,d)$ compact $\Rightarrow$ $diam(X) < \infty$

and must be attained

$\exists x, y \in X$ $d(x,y) = diam(X)$

Ex. $\mathbb{R}$ is not compact (with Eucl. dist)

$(0,1)$ not compact

Ex. $\mathbb{R}$, $\bar{d} = \min(d, 1)$ bounded & attains diam but same top as $d$ $\Rightarrow$ not compact

Ex. $\mathbb{R}$ with discrete dist is bounded and attains diameter but not compact

(every conv. seq. must be eventually constant)

But $[0,1]$ is compact (with Eucl. dist)

prove now!

Lemma $x_n \subset \mathbb{R}$ increasing $x_n \geq x_{n-1}$

strictly $>$

de $\leq$

s. de. $<$

monotonous = increasing or decreasing

**Lemma** $x_n$ increasing & bounded (above)

$$x_n \to \sup\{x_n\}$$

Similarly decreasing

**Lemma** $(x_n)$ bounded $\Rightarrow \exists (x_{n_m})$ monotonous.

**Pf.** Assume $\nexists (x_{n_m})$ increasing. will show th. by finding one decreasing subseq

$n_0 = 0$ $k = 0$

let $X = \sup\{x_n : n > n_k\}$

if $\infty$ly many $x_n = X$ $\exists$ constant subseq $\Rightarrow$ incr. ↯

if ex. no. $x_n = X$

$\exists n_1$ $x_{n_1} > X - 1$ $x_{n_1} < X$

$\max\{x_1, \dots, x_{n_1}\} < X$

$\exists n_2 > n_1$ $x_{n_2} > x_{n_1}$ $x_{n_2} < X$

$x_{n_i}$ increasing ↯

so $\exists$ finitely many $x_n = X$

let $n_{k+1} = \max\{n : x_n = X\}$ $x_{n'} < x_{n_{k+1}} \; (= X)$ $n' > n_{k+1}$

$x_{n_k}$ decreasing ■

**Th** (Bolzano-Weierstrass) Every bounded seq in $\mathbb{R}$ $\supset$ conv. subsequence ■

Ex. $[0,1]$ $(x_n) \subset [0,1]$

$x_n \subset \mathbb{R}$ bounded $\Rightarrow x_{n_m} \to x$

$[0,1]$ closed $\Rightarrow x \in [0,1]$

$x_{n_m} \to x \in [0,1]$ in relative top.

$\Rightarrow [0,1]$ compact.

Th. $(X_1, d_1)$, $(X_2, d_2)$ compact

$\Rightarrow (X_1 \times X_2, \underset{\uparrow}{d_1 + d_2})$ compact

or any d inducing product top.

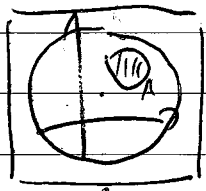
$\sqrt{d_1^2 + d_2^2}$ ....

Th. $(X, d)$ compact $A \subset X$ compact (in rel. top.)

$\Leftrightarrow A = \overline{A}$ $\quad d|_{A \times A}$

Th. $(X, d)$ any metric space $A \subset X$ compact

$\Rightarrow A$ closed and bounded $\quad A \subset B \subset I \times I'$



Th. $X = \mathbb{R}^n$ with Euclidean metric

(Heine-Borel)

$A$ closed & bounded $\Leftrightarrow$ compact

Ex. $[0,1]^\omega \subset \mathbb{R}^\omega$ with uniform metric closed & bounded but $\exists f = \{ {1 \atop 0} {x = n \atop x \neq n} \}$ $\nexists$ uniformly

Th: $f : (X, d) \to \mathbb{R}$ continuous $X$ compact

$f$ bounded & $f$ attains max. value

Pf. If $\exists f(x_n) \to \infty$ $x_n \in X$

$\exists x_{n_m} \subset x_n$ conv. $x_{n_m} \to x$

f continuous $f(x_{n_m}) \to f(x)$ ↯ $\in \mathbb{R}$.

So assume $f(x_n)$ bounded.

$\forall \varepsilon > 0$ $\exists x_\varepsilon \in X$ $f(x_\varepsilon) > \sup(f(X)) - \varepsilon$

$(x_{1/n})$ has conv. subseq $x_{n_m} \to x \in X$

$f(x_{n_m}) \to f(x)$

$\downarrow$ $\sup f$

$f(x) = \sup f = \max f$.

Cor $(X,d)$ compact. $X$ attains diam $< \infty$

Pf. $d: \underbrace{X \times X}_{\text{compact}} \to \mathbb{R}$ continuous. $\square$

↑ 2.20.3 p. 124 Thm 4

!! Def. $(X,d)$ is ~~limit~~ acc. pt compact ← I use not to confuse with my use of "limit pt"

$\iff \forall S \subset X$ $|S| = \infty$ $S$ has acc. pt. $A_{acc} \neq \emptyset$

~~Thm~~ Lemma $(X,d)$ acc pt compact $\iff$ seq. compact

Pf. $\Leftarrow$ $|S| = \infty$ $\exists s_1, s_2 \in S$ $s_i \neq s_j$

$\exists s_{n_m} \to x$ at most one $s_{n_m} = x$

$\cap$ $S$ $x$ acc. pt of $S$

$\Longrightarrow$ $(x_n) \subset X$ seq. if set $\{x_n\}$ finite, $\exists$ only many equal $x_{n_m}$ $\to$ done

So $|\{x_n\}|$ inf. it has acc. pt. $x$, choose $x_{n_m} \in B_{1/m}(x)$, $x_{n_m} \to x$

(79)

**Theo** The ordered square $[0,1]^2$ with the order top. of the lexicogr. order is not metrizable (but it is $T_4$ like any order top.; it is also linear cont.)

promised on (76)

promised Theo as appl. of comp.

**Pf.** assume $[0,1]^2$ were metrizable

define $f : [0,1] \to \mathbb{R}$ by $f(x) = \operatorname{diam}(\{x\} \times [0,1]) > 0$

(w.l.o.g. bound the metric to avoid $\infty$)

$\operatorname{diam}(A) = \sup\{d(x,y) : x,y \in A\}$

Consider $x_0 \in [0,1]$. Let $x_n \downarrow x_0$. Then $(x_n, 1) \to (x_0, 1)$

and in fact for any seq $x_n' \in [\underbrace{(x_0, 1), (x_n, 1)}_{D_n}]$

$x_n' \to (x_0, 1)$

thus $\lim_{n\to\infty} \bar{d}((x_0,1), D_n) = 0$

$\bar{d}(x, A) = \sup\{d(x,y) : y \in A\}$

$\operatorname{diam}(D_n) \le 2\bar{d}((x_0,1), D_n)$ by $\triangle$

so $\operatorname{diam}(D_n) \to 0$ and $D_n \supset \{x_n\} \times [0,1]$

$\Longrightarrow f(x_n) = \operatorname{diam}(\{x_n\} \times [0,1]) \to 0$

So $\lim_{x \to x_0^+} f(x) = 0$.

Similarly let $x_n \nearrow x_0$ argue with $(x_0, 0) < (x_n, 0)$

that $\lim_{x \to x_0^-} f(x) = 0$. Thus now

$\forall x_0 \in [0,1]$ $\quad \lim_{x \to x_0} f(x) = 0$, but $f(x_0) > 0$.

Claim $\nexists$ such $f$. $\Rightarrow$ ⊠

Proof: Let $\varepsilon > 0$.

If $\exists$ $\infty$ly many $x \in [0,1]$ with $f(x) \geq \varepsilon$
then by compactness they have an acc. pt.
$\exists\, x_n \to x_0 \in [0,1]$ $\quad x_0 \neq x_n$
$f(x_n) \geq \varepsilon$ $\quad$ ↯ to $\lim_{x \to x_0} f(x) = 0$.

So $\forall \varepsilon$ $S_\varepsilon = |\{x : f(x) \geq \varepsilon\}| < \infty$

Then $S = |\{x : f(x) > 0\}| = \bigcup_{i=1}^{\infty} S_{1/n}$ is countable,

but we wanted $S = [0,1]$, which is uncountable. ↯ ⊠

## Covering compactness §26

HW do it for $[0,1] \times [0,1]$

Similar proof for
$\{f : \mathbb{R} \to \mathbb{R}\}$ product top.

but $\{f : \mathbb{Z}_+ \to \mathbb{R}\} = \mathbb{R}^\omega$ with
product top is m'ble Th 20.5
p. 123

Def: $(X,d)$ metric space
$\mathcal{O} \subset \mathcal{P}(X)$ is an open cover(ing) of $X$ if
$\bigcup \mathcal{O} = X$ and $\forall O \in \mathcal{O}$ $O$ open in $X$

$\mathcal{O}' \subset \mathcal{P}(X)$ is called a subcover of $\mathcal{O}$ if $\mathcal{O}' \subset \mathcal{O}$ and
$\mathcal{O}'$ is a cover

Def: $X$ is covering compact if every open cover of $X$ contains a finite subcover

Th (the Borel-Lebesgue covering theo)

$(X,d)$ metr space $X$ seq compact $\Leftrightarrow$ $X$ covering compact.

for the proof we need lemmas

Lem $(X,d)$ seq. compact $\Rightarrow \forall\ X_1 \supset X_2 \supset X_3 \supset \dots$ filtration with non-empty closed sets, we have $\bigcap_{i=1}^{\infty} X_i \neq \emptyset$.

Pf. Take $x_i \in X_i$ $\exists\, x_{n_i} \to x$ $X_i$ closed $x \in X_i\ \forall i$ ▫

Rem $\Leftarrow$ also true see last part of pf of B-L below, (84)

Ex. $X = \mathbb{R}$ $X_i = (-\infty, -i]$ closed filtration $\bigcap X_i = \emptyset$

Lemma (HW) $x_n \to x$ in $(X,d)$ $\Leftrightarrow$ $\forall \varepsilon > 0\ \exists N\ \forall n,m \geq N\ d(x_n, x_m) < \varepsilon$ (Cauchy-prop) ▫

Proof of B-L $\Rightarrow$ Prof. Friedrich 7.11.90

Step 1 $X$ seq compact $\Rightarrow \exists$ finite $\varepsilon$-net
Step 2 $\Rightarrow X$ separable (HW)
Step 3 $\Rightarrow X$ is Lindelöf (every cover has countable subcover)
Step 4 $\Rightarrow X$ covering compact

Step 1. $X$ (seq.) compact. $\Rightarrow$

$\forall \varepsilon > 0\ \exists p_1, \dots, p_k \in X$

$\bigcup_{i=1}^{k} B_\varepsilon(p_i) = X.$

$\{p_1, \dots, p_k\}$ $\varepsilon$-net

Pf. by ~~y~~ assume $\exists \varepsilon > 0\ \forall p_1, \dots, p_k\ \forall k$

$\bigcup_{i=1}^{k} B_\varepsilon(p_i) \subsetneq X$

take $p_0 \in X$ $B_\varepsilon(p_0) \subsetneq X$ $\exists p_1 \notin B_\varepsilon(p_0)$.

$B_\varepsilon(p_0) \cup B_\varepsilon(p_1) \not\subseteq X$

$\exists p_2 \notin$

so find a seq $p_i \subset X$ with $d(p_i, p_j) > \varepsilon$ $\forall i,j$

any subseq gives $\nrightarrow$ to Cauchy-prop $\Rightarrow$ does not converge.

<u>Step 2.</u> $X$. (seq.) compact $\Rightarrow \exists A \subset X$ $|A| \le \omega$, $\bar{A} = X$

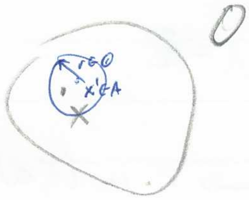
X separable

$A = \bigcup_{n=1}^{\infty} \{ \text{finite } 1/n\text{-net} \}$ (think about it!) used $d(x,A)$ lemma

<u>Step 3</u> $X$ compact $\mathcal{O}$ open cover

$\mathcal{O} \supset \mathcal{O}'$ countable subcover

let $A \subset X$, $|A| \le \omega$, $\bar{A} = X$.




$\Lambda = \{ (a,r) \in A \times \mathbb{Q} : \overset{\exists O \in \mathcal{O}}{B_r(a) \subset O} \}$

$\Lambda$ countable

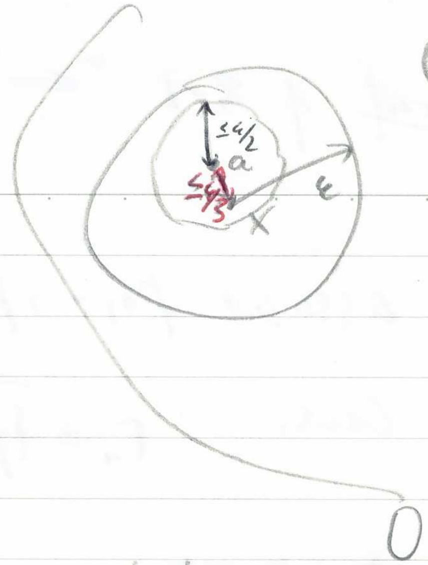
for each $(a,r) \in \Lambda$ choose an $O = O(a,r) \in \mathcal{O}$ with $B_r(a) \subset O$ (AC!)

$\mathcal{O}' = \{ O(a,r) \; (a,r) \in \Lambda \}$ $|\mathcal{O}'| \le \omega$

<u>claim</u> $\mathcal{O}'$ is open cover of $X$ (subcover of $\mathcal{O}$), i.e., $\bigcup \mathcal{O}' = X$

<u>Pf.</u> $x \in X$ $\mathcal{O}$ cover $\exists O \in \mathcal{O}$ $x \in O$

$\exists \varepsilon > 0$ $B_\varepsilon(x) \subset O$




since $A$ is dense

$\exists a \in A \quad d(x,a) < \varepsilon/3$

$\Rightarrow$ take $r \in \mathbb{Q} \cap (\varepsilon/3, \varepsilon/2)$

$r > \varepsilon/3 \quad B_r(a) \ni x$

$r < \varepsilon/2 \quad B_r(a) \subset B_{\varepsilon/3 + \varepsilon/2}(x)$

$\subset B_\varepsilon(x) \subset O$

$\exists O \in \mathcal{O} \quad B_r(a) \subset O$

$(a,r) \in \Lambda \quad x \in B_r(a) \subset O' = O(a,r) \in \mathcal{O}'$

$\forall x \in X \quad \exists O' \in \mathcal{O}' \quad x \in O'$. $\square$

<u>Step 4</u> (final) $\mathcal{O}$ open cover

$\exists \mathcal{O}'$ countable subcover $X = O_1 \cup O_2 \cup \dots \cup \dots$

Consider $F_n = X \setminus \bigcup_{i=1}^{n} O_i$

$F_1 \supset F_2 \supset \dots \quad F_i$ closed $\quad$ if $\forall i \; F_i \neq \emptyset$

$\underset{\text{Lem}}{\Rightarrow} \bigcap_{i=1}^{\infty} F_i \neq \emptyset \Rightarrow \bigcup_{i=1}^{\infty} O_i \neq X$ ↯

so some $F_n = \emptyset \Rightarrow X = \bigcup_{i=1}^{n} O_i$

$\mathcal{O}' = \{O_i\}_{i=1}^{n}$ is finite subcover of $\mathcal{O}$. $\square$

(end of "$\Rightarrow$")

Proof of B-L "$\Leftarrow$" X covering comp $\Rightarrow$ X seq. compact.

assume $p_1, \dots, p_n, \dots$ seq in $X$.

Cons. $F_n = \overline{\{p_{n+1}, p_{n+2}, \dots\}}$ $\quad U_n = X \setminus F_n$

$F_1 \supset F_2 \supset F_3$ closed $\qquad U_1 \subset U_2 \subset U_3$ open

$$\text{If } \bigcup_{n=1}^{\infty} U_n = X \underset{\text{cover prop}}{\Longrightarrow} \bigcup_{n=1}^{N} U_n = X$$

$$\to F_N = \emptyset \ \text{ⓔ}$$

Thus $\bigcup_{n=1}^{\infty} U_i \neq X \Rightarrow \bigcap_{n=1}^{\infty} F_i \neq \emptyset.$

$p \in \bigcap_{n=1}^{\infty} F_i \qquad p \in F_1 = \overline{\{p_2, p_3, \dots\}}$

$\varepsilon = 1 \quad \exists n_1 > 1 \quad d(p_{n_1}, p) < 1$

$p \in F_{n_1} = \overline{\{p_{n_1+1}, \dots, \}}$

$\varepsilon = 1/2 \quad \exists n_2 > n_1 \quad d(p_{n_2}, p) < 1/2$

$p \in F_{n_2} = \overline{\{p_{n_2+1}, \dots\}}$

$\varepsilon = 1/3 \quad \exists n_3 > n_2 \quad d(p_{n_3}, p) < 1/3$

$\dots \quad p_{n_i} \to p$

subseq of $(p_n)$

$\boxed{\text{Dini theorem! 27.6 p174}}$

Now to general top. spaces

covering compactness §26
limit pt. compactness §28

$$\begin{array}{ccc} \text{covering comp.} & \overset{\not\Leftarrow \text{ Th. 28.1 p. 177 (5}\Omega)}{\Longrightarrow} & \text{acc. pt compactness} \\ \times (85a) \not\Downarrow \Uparrow \times S_\Omega & \nearrow \text{ works as above} & \not\swarrow \times (85a) \\ \text{seq. compactness} & & \end{array}$$

<u>lemma</u> order top is $\overset{\text{covering}}{\text{compact}}$ $\Rightarrow$ $\exists$ smallest & largest el.

<u>Th</u> order top s.t. $\exists$ smallest, largest el. & LUBP $\Rightarrow$ $\underset{\text{seq.}}{\text{compact}}$ (gen. Bolzano-W.)

<u>Ex.</u> $S_\Omega = \bar{S}_\Omega \setminus \{\Omega\}$ "smallest uncountable ordered set"

has no largest el $\Rightarrow$ not $\overset{\text{covering}}{\text{compact}}$

but $S_\Omega$ is seq. (and hence acc. pt.) compact $\Rightarrow$ not metrizable!

$x_1, x_2, \ldots \subset S_\Omega$ $\{x_i\} \subset S_\Omega$ $|\{x_i\}| \le \omega$ $\Rightarrow$ $\{x_i\}$ bounded

$\exists b \in S_\Omega$ $\{x_i\} \subset [a_0, b]$ seq. compact

$a_0$ smallest el. of $S_\Omega$

gen. B-W. $\Rightarrow$ $\exists$ conv. subseq. in $[a_0, b]$ $\Rightarrow$ also conv. in $S_\Omega$

(order top. of interval = rel. top.)

<u>Rem</u> This ex. shows also if $x \in \bar{A}$ (57) then not necessarily $x \in A_{\lim}$ $\Leftarrow$ $A = S_\Omega$, $\hat{X} = \bar{S}_\Omega$, $x \in \Omega$

like on (70) $\bar{A} = \bar{S}_\Omega$ but $\Omega \notin (S_\Omega)_{\lim}$ $\leftarrow$ explains the notation $\bar{S}_\Omega$

**Th.** Tychonoff's theo (§37, p.167)

The product topology of (any # of) (covering) compact spaces is (covering) compact

**Ex.** $\Sigma = \{0,1\}^{\mathbb{Z}_+} = \{\text{seq. of } 0,1\}$ ($\{0,1\}$ discr. top.)

$X = \{0,1\}^{\Sigma}_{prod}$ is (c.)compact by Tychonoff's theo

($\Rightarrow$ also acc. pt. compact) ($X$ is $T_2$ etc.)

but $X$ is not seq. compact

**Pf.** $\underset{\{0,1\}^{\Sigma}}{f_n} \longrightarrow \underset{\{0,1\}^{\Sigma}}{g}$ $\forall \sigma \in \Sigma$ $\exists n$ $f_i(\sigma) = g(\sigma)$ $i \geq n$

ptwise convergence

$\Rightarrow$ $(f_n)$ conv $\Leftrightarrow$ $\forall \sigma \in \Sigma$ $\{f_n(\sigma)\}$ is eventually constant

Cons $(f_n) \subset \{0,1\}^{\Sigma}$ given by $f_n(\sigma) = \sigma(n)$

$f_n : \Sigma \to \{0,1\}$ $f_n((0,1,0,\dots,\underset{n}{0,1},1,0\dots)) = 1$

let $(f_{n_k}) \subset (f_n)$

choose $\hat{\sigma} \in \Sigma$ with $\hat{\sigma}(n_i) = \left\{ \begin{matrix} 1 & ; \text{ even} \\ 0 & ; \text{ odd} \end{matrix} \right\}$

$f_{n_k}(\hat{\sigma}) = \hat{\sigma}(n_k) = \left\{ \begin{matrix} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{matrix} \right\}$ is not eventually const

$f_{n_k} \not\to g$

Rem: there are other exs like $X_{disc} \times \{0,1\}_{indisc}$ $|X| = \infty$

but not even $T_0$

85a has an ex of a space that is covering (and acc. pt. l.c., but not seq compact,

but this ex uses Tychonoff's theorem

but recall this example used to 70 ↓

It took me some to give an ex.

If you like ↓ 70

of $A \subset X$ with

$\overline{A} \setminus A \neq \emptyset$ but $A_{\lim} = \emptyset$, i.e.

A not closed ($\Rightarrow$ has acc pt) but no converging seq (exept constant)

$A_{acc} \neq \emptyset$

**Ex.** Cons $X = \{ f : \mathbb{R} \to \mathbb{R}_{discr} \}$ with range $\mathbb{R}$ with discrete top

(may skip) not very good

and pt wise con. top $X = \mathcal{F}(\mathbb{R}, \mathbb{R}_{dis})_{prod}$

(product discrete top.)

i.e. $f_n \to f \iff \forall x \in \mathbb{R}\ \exists N\ \forall n \geq N$ $f_n(x) = f(x)$

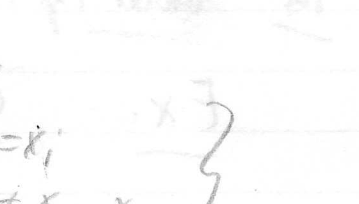
the basis of the set

$$\left\{ U_{S, y_1, \dots, y_n} = \prod_{x \in \mathbb{R}} \begin{cases} \{y_i\} & x = x_i \\ \mathbb{R} & x \neq x_1, \dots, x_n \end{cases} \ : \ S = \{x_1, \dots, x_n\} \subset \mathbb{R} \text{ finite } (x_i \neq x_j),\ y_1, \dots, y_n \in \mathbb{R} \right\}$$

Fix a bijection $\Phi : \{\text{finite sub sets of } \mathbb{R}\} \to \mathbb{R} \setminus \{0\}$

Cons the family

$$f_S(x) = \begin{cases} 0 & x \in S \\ \Phi(S) & x \notin S \end{cases}$$




constant fct except for finite zero set.

$A = \{ f_S : S \subset \mathbb{R} \text{ finite} \} \subset X$

87

fct. ($f(x)=0$ $\forall x \in \mathbb{R}$)

$0 \in \bar{A} \setminus A$ A has accumulation pt 0

If $0 \in U$ open U basis el.

$U = U_{S,\underbrace{0,\dots,0}_{y_i=0}}$ for some $S \subset \mathbb{R}$ $|S| < \infty$

$f_S \in U \cap A \neq \emptyset$ $\Rightarrow$ $0 \in \bar{A}$ obv. $0 \notin A$.

Now take sequence $f_{S_1}, f_{S_2}, \dots$ in A $f_{S_i} \neq f_{S_j}$ $i \neq j$

$\bigcup_{i=1}^{\infty} S_i$ is countable take $x_0 \in \mathbb{R} \setminus \bigcup_{i=1}^{\infty} S_i$

$f_{S_n}(x_0) = \mathbb{1}_{S_n}$ $f_{S_n}(x_0) \neq f_{S_m}(x_0)$ $\forall n, m \in \mathbb{Z}_+$ $n \neq m$

$f_{S_n}(x_0)$ does not converge in discr. top

$\Rightarrow f_{S_n} \not\to f$

So A contains no conv. sequence!

(except eventually constant)

($\Rightarrow$ X not metrizable etc.)

End Ex. □

Note: $A_{\lim} = \emptyset$ means A has no limit pt in X not in A .. A has no limit pt in A just means A is discrete, and I can get you discrete non-closed sets in simple spaces like $A = \{\frac{1}{n} : n \in \mathbb{Z}_+\} \subset \mathbb{R} = X$

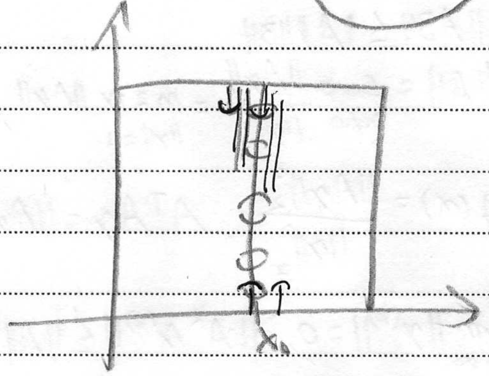
Pb Prove that $I_o^2$ ordered square is covering compact.

~~4/~~

Sol $\mathcal{O}$ open cover.

let $x_0 \in [0,1]$

$I(x_0) = x_0 \times [0,1]$



$\mathcal{O}$ induces covering of $I(x_0)$

~~2/~~ 0/15

$\tilde{\mathcal{O}}_{x_0} = \{ O \cap I(x_0) : O \in \mathcal{O} \}$

$I(x_0) \cong [0,1]$ (covering) comp. ~~$\Rightarrow \exists \mathcal{O}_{x_0} \subset \mathcal{O} \quad \cup \mathcal{O}_{x_0} \supset I(x_0)$~~

$\exists \tilde{\mathcal{O}}'_{x_0} \subset \tilde{\mathcal{O}}_{x_0} \quad |\tilde{\mathcal{O}}'_{x_0}| < \infty \quad \cup \tilde{\mathcal{O}}'_{x_0} = I(x_0)$ ~~$\mathcal{O}_{x_0} = \{ O \in \mathcal{O} :$~~

for each $\tilde{O}' \in \tilde{\mathcal{O}}'_{x_0}$ choose

an $\hat{O}_{\tilde{O}'} \in \mathcal{O}$ with $\hat{O}_{\tilde{O}'} \cap I(x_0) = \tilde{O}'$

$\mathcal{O}_{x_0} = \{ \hat{O}_{\tilde{O}'} : \tilde{O}' \in \tilde{\mathcal{O}}'_{x_0} \}$

$|\mathcal{O}_{x_0}| < \infty$

$\Rightarrow \exists \mathcal{O}_{x_0} \subset \mathcal{O} \quad |\mathcal{O}_{x_0}| < \infty \quad \underbrace{\cup \mathcal{O}_{x_0}}_{\Lambda_{x_0}} \supset I(x_0)$

$\Lambda_{x_0} \ni x_0 \times 0$ open $\Rightarrow \Lambda_{x_0} \supset (x_0' \times x_0', x_0 \times 0] \quad x_0' < x_0$ $(x_0 \neq 0)$

$\Rightarrow \Lambda_{x_0} \supset (x_0', x_0) \times [0,1]$

similarly if $x_0 \neq 1$ $\Lambda_{x_0} \ni x_0 \times 1$

$\Rightarrow \Lambda_{x_0} \supset (x_0, x_0'') \times [0,1] \quad x_0'' > x_0$

$\Rightarrow \Lambda_{x_0} \supset (x_0', x_0'') \times [0,1]$

87b

so for each $x_0 \in [0,1]$

$\exists$ finite subcover of $\mathcal{O}$, $\mathcal{O}_{x_0}$ ~~covering~~ and

$\exists\ x_0' < x_0 < x_0''$ . $\bigcup \mathcal{O}_{x_0} \supset (x_0', x_0'') \cap [0,1]$

~~$x_0'' = x_0$ / $x_0 =$~~ with $(x_0', x_0'') = (x_0', 1]$ $x_0 = 1$

$= [0, x_0'')$ $x_0 = 0$

Now $\{ (x_0', x_0'')\ \ x_0 \in [0,1] \}$ is an open cover

of $[0,1]$

$[0,1]$ comp. $\Rightarrow$ $\exists\ x_1, \dots, x_n$ : $(x_1', x_1'') \cup \dots \cup (x_n', x_n'') = [0,1]$

then $\bigcup_{i=1}^{n} \mathcal{O}_{x_i}$ is a finite subcover of $\mathcal{O}$. ∎

Pb Prove that $I_0^2$ is seq. compact.

Pf. Let $(x_n \times y_n) \subset I_0^2$.

$(x_n) \subset [0,1]$ cy. $\exists\ x_{n_k} \to x$

w.l.o.g. (1) $x_{n_k} \equiv x$ (2) $x_{n_k} \nearrow x$ (3) $x_{n_k} \searrow x$

go over to subseq.

(2) $(x_{n_k}, y_{n_k}) \to (x, 0)$ (3) $(x_{n_k}, y_{n_k}) \to (x, 1)$

(1) $\exists (y_{n_{k_l}}) \subset (y_{n_k})$ $y_{n_{k_l}} \to y$. $(x_{n_{k_l}}, y_{n_{k_l}}) \to (x, y)$ ∎

No.

Date. / /

(87c)

the $l_0^2$ is seq. compact.

[10] c) quadrize this arg to prove the $[0,1]^\omega$ is seq compact with the dict order

I haven't been able to figure out yet if it is seq compact.

Sol. Assume $(f_k)_{k=1}^{\infty} \in \mathcal{F}(\mathbb{Z}_+, [0,1]) \cong [0,1]^\omega$ does not have conv subseq in dict order.

We construct for each $n \in \mathbb{Z}_+$ a subseq $(f_{k,n})_{k=1}^{\infty}$ of $f_n$ and a seq $(c_k)_{k=1}^{\infty}$ such that $(f_{k,n+1}) \subset (f_{k,n})$ is a subseq

$f_{k,n} |_{\{1,\dots,n-1\}} = c|_{\{1,\dots,n-1\}}$ and then

By ind $i=1$ ✓ $f_{k,1} = f_k$

Ass $(f_{k,n})_{k=1}^{\infty}$ constructed as well as $c_1, \dots, c_{n-1}$

Cons. If $(f_{k,n}(n))_{k=1}^{\infty}$ w.l.o.g. $\exists (f_{k,n+1}) \subset (f_k)$

$f_{k,n+1}(n) \to c_n$ ~~s.t.~~ either (1) $f_{k,n+1}(n) \equiv c_n$ $(k \to \infty)$

w.l.o.g.

(2) $f_{k,n+1}(n) \nearrow c_n$ $(k \to \infty)$

(3) $f_{k,n+1}(n) \searrow c_n$ $(k \to \infty)$

(2) $f_{k,n+1} \xrightarrow[k \to \infty]{} f(\ell) = \begin{cases} c_\ell & \ell \le n \\ 0 & \ell > n \end{cases}$ ↯

(3) $f_{k,n+1} \xrightarrow[k \to \infty]{} f(\ell) = \begin{cases} c_\ell & \ell \le n \\ 1 & \ell > n \end{cases}$ ↯

(1) constr. $(f_{k,n+1})$ and $c_n$ ind complete.

So now $\exists\, (f_{k,n}) \subset (f_k)$ , $(c_1, c_2, \dots)$ as needed.

cons. $(f_{k,k})_{k=1}^{\infty}$ diag'tion.

87d

$f_{k,k}(n) = c_n \quad k \geq n$

$f_{k,k} \xrightarrow[k\to\infty]{} (c_1, c_2, c_3, \dots)$ ↯. ☐

$\overset{12}{\{f(n) = c_n\}}$

Thus for non-metrizable spaces '(any sort of) compact $\Rightarrow$ separable' is false!