§23 Connected spaces

Def: X top. space   separation (I write (disjoint) decomp)

U, V $U, V \subset X$ open $U, V \neq \emptyset$

$U \cup V = X$ $U \cap V = \emptyset$

X connected $\Leftrightarrow$ X has no separation X discon $\Leftrightarrow \exists$ separation

Rem: X is connected $\Leftrightarrow$ $\forall A \subset X$ A open & closed $\Rightarrow A = \emptyset$ or $A = X$

Rem: do not use "separable" in this context for disconnected : separ. $\exists A \subset X$ $\bar{A} = X$ $|A| \leq \omega$ (82)

Def: $A \subset X$ separation of A in X

U, V $U, V \neq \emptyset$ U, V not nec. open

$A = U \cup V$ $U \cap \bar{V} = \emptyset$, $\bar{U} \cap V = \emptyset$ U, V disjoint and none contains an acc. point of the other.

Lemma A is conn. (in rel. top.) $\Leftrightarrow$ A has no sep.

Ex. $X = \mathbb{R}$ $A = \underbrace{[-1, 0)}_{U} \cup \underbrace{(0, 1]}_{V}$

$$\overline{(0,1]} \cap [-1,0) = [0,1] \cap [-1,0) = \emptyset$$
$$(0,1] \cap \overline{[-1,0)} = (0,1] \cap [-1,0] = \emptyset$$

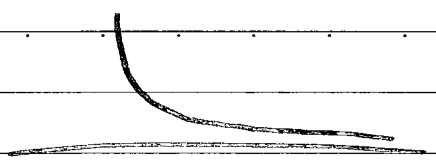
A disconnected

Rem: $\bar{U} \cap \bar{V} = \{0\} \neq \emptyset$ but this is not forbidden!

$\uparrow$

U, V have common accumulation pt

Ex. $\mathbb{R}^2$

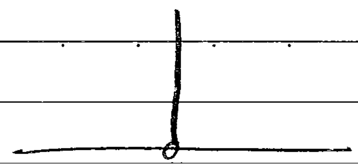


$\{(x,0) : x\in\mathbb{R}\} \cup \{(x,1/x)\ x>0\}$

separation

$U \cap V = \bar{U} \cap \bar{V} = \emptyset$

Ex $\mathbb{R}^2$



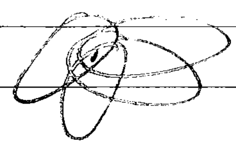
$\underbrace{\{(x,0) : x\in\mathbb{R}\}}_{U} \cup \underbrace{\{(0,1/x)\ x>0\}}_{V}$

no separation

$\bar{V} \cap U \neq \emptyset$

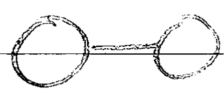
Lemma: C, D separation of X, $Y\subset X$ connected
$\Rightarrow Y \subset C$ or $Y \subset D$

Th: $X \ni x$, $\mathcal{Y} \subset \mathcal{P}(X)$, $\forall Y \in \mathcal{Y}$ Y conn., $Y \ni x$
$\Rightarrow \bigcup \mathcal{Y}$ conn'd



Th. If $A \subset X$ conn. and $A \subset B \subset \bar{A}$
$\Rightarrow B$ connected

Ex. not true for interior!



A

Th. The image of a conn'd set under cont's map is conn'd.

Th. X, Y conn'd $\Rightarrow X \times Y$ conn'd

$\Rightarrow$ true for finite products

Ex. $\mathbb{R}^\omega$ box top. or uniform top.

$\mathbb{R}$ conn'd (prove later)

$\mathbb{R}^\omega = \{$ bounded seq. $\} \cup \{$ unbound. seq $\}$

both open and disjoint non-empty $\Rightarrow \mathbb{R}^\omega$ disconn. with unif. & box top

Ex. $\mathbb{R}^\omega$ with product top.

$\mathbb{R}^n \simeq \{(x_1, \dots, x_n, 0, \dots, 0)\} \hookrightarrow \mathbb{R}^\omega$

conn'd

$\bigcap \mathbb{R}^n = 0 \quad R = \bigcup_{n=1}^{\infty} \mathbb{R}^n$ conn'd

$\mathbb{R}^\omega = \overline{R} \qquad f : \mathbb{Z}_+ \to \mathbb{R} \quad \in \mathbb{R}^\omega$

R dense $\qquad f_n \begin{cases} f(x) & x \le n \\ 0 & x > n \end{cases} \to f$ pointwise

$\mathbb{R}^\omega$ conn'd

§24 Connected Subspaces of $\mathbb{R}$

Def. (recall) (15 bottom) $(X, <)$ ordered set is a linear continuum if

0) $|L| > 1$

1) LUBP

2) $\forall x, y \in X \quad x < y \Rightarrow \exists z \in X \quad x < z < y$

Intermediate Element Property (IEP)

Th. L is linear continuum w/ order top, $Y \subset L$ convex $\Rightarrow Y$ conn. t.p.l connected, also are intervals and rays in L

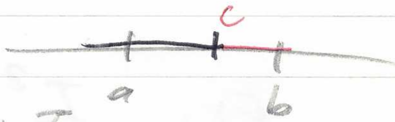
recall: $Y \subset L$ convex $\forall x,y \in Y$ $x<y$ $\forall z \in L$ $x<z<y \Rightarrow z \in Y$ (91)

Pf: By ↯

assume $Y = A \cup B$ $A \cap B = \emptyset$ $A, B \neq \emptyset$ A, B open

$a \in A$ $b \in B$ assume w.l.o.g. $a<b$

$I = [a,b] = \underset{A_0}{A \cap [a,b]} \cup \underset{B_0}{B \cap [a,b]}$

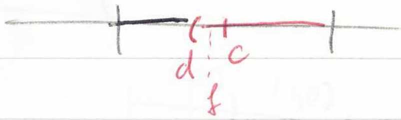


A_0, B_0 open in I

$c = \sup \underbrace{A \cap [a,b]}_{A_0}$

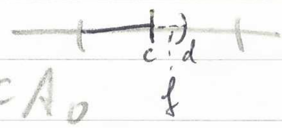
1) If $c \in B_0 \Rightarrow c \neq a \Rightarrow \exists d<c$ $(d,c] \subset B_0$

B_0 open in I



$\exists d<f<c$ $[f,c] \subset B_0$

$[f,c] \cap A_0 = \emptyset$ ↯ to $c = \sup A_0$



2) If $c \in A_0 \Rightarrow c \neq b \Rightarrow \exists [c,d) \subset A_0$

A_0 open in I

$\exists c<f<d$ $f \in A_0$ ↯ to $c = \sup A_0$ ∎

Cor: $\mathbb{R}$ conn'd and intervals and rays in it.

Rem: converse of Th also true: $(X, \mathcal{A}_<)$ conn. $\Rightarrow$ $(X,<)$ is lin. cont.

Th: Intermediate value Th.

$(X, \mathcal{A})$ conn'd. $(Y,<)$ order top.

$f: X \to Y$ cont's $\Rightarrow$ $\forall x,y \in X$ $\forall c \in [f(x), f(y)]$

$\exists d \in X$ $f(d) = c$

Pf. by ↯ if $\exists r \in (f(x), f(y))$ $r \notin f(X)$

Con $X = f^{-1}((-\infty, r)) \cup f^{-1}((r, \infty))$

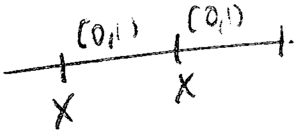
separation of X

Rem: take $X' = [x,y] \Rightarrow \exists d \in [x,y]$
"Darboux property"

Ex. I_0^2 ordered square with dic order
is a lin. continuum diff from $\mathbb{R}$

Ex. X well ordered $X \times [0,1)$ with dict order
is a linear continuum

important case:



$X = S_\Omega \ (= \bar{S}_\Omega \setminus \{\Omega\})$

consider $S_\Omega \times [0,1) \setminus \{a_0 \times 0\} =: L$

with dict order

topologists' long line

L is locally homeomorphic to $\mathbb{R}$
every pt has a neighborhood homeomorphic
to an interval

but not embeddable in $\mathbb{R}$
it is not separable

§25 Components

Def: X top. space define equiv rel on
X by $x, y \in X$

$x \sim y$ $\exists$ connected subspace of X $W \ni x, y$

Def. $[x]_\sim$ is the (eq. class) connected comp of $x \in X$

$C_x = [x]_\sim = \bigcup \{ C \subset X \text{ conn. } C \ni x \}$ connected by (89) max conn. set $\ni x$

Rem. C_x closed as $\overline{C_x} \ni x$ is conn'd. $\Rightarrow \overline{C_x} \subset C_x$

Def: X satisfies the connected nghb cond (CNC) (at $\tilde{x}$) if every (of $\tilde{x}$) pt has a conn'd neighborhood.

Lemma $y \in C_x \Longrightarrow \forall (U,V)$ separ. of X $x \in U \Rightarrow y \in U$

$\Longleftarrow$ if CNC see Ex 3.26.10

Pf. $\Longrightarrow$ $\exists$ conn. subsp. $W \ni x, y$

assume by [illegible] $\exists (U,V)$ $x \in U$ $y \in V$

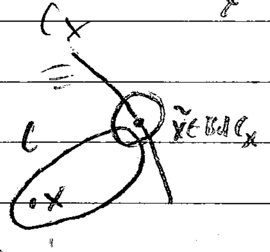
$(U \cap W, V \cap W)$ sep. of W

W conn. $\Rightarrow \underset{x}{U \cap W} = \emptyset$ or $\underset{y}{V \cap W} = \emptyset$ ↯

$\Longleftarrow$ $\neg x \sim y$

$\forall W \ni x$ conn. $W \not\ni y$.

$C_x = \bigcup \{ W \ni x \text{ conn} \} \not\ni y$

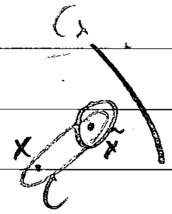


(2)

W conn. neighborhood cond $\Rightarrow C_x$ open (and closed)
(1) (2) or Rem above

$\Longrightarrow \underset{x}{C_x} \cup \underset{y}{(X \setminus C_x)}$

is a separation



(1)

Note: If $A \subset X$ conn. comp of A is meant w.r.t. relative top.

Def: X is totally disconnected if all its conn. comps are points

Rem $A \subset X$ discrete $\Rightarrow$ A totally disc. (as rel. top.)

converse is false

Ex. $\mathbb{Q} \subset \mathbb{R}$ is totally disconn. (but certainly not discrete)

$x<y \quad \exists\, r \in (x,y)$ irr.

$\mathbb{Q} = (\mathbb{Q} \cap (-\infty, r)) \cup (\mathbb{Q} \cap (r, \infty))$ open in $\mathbb{Q}$

I can separate pts by open sets (whose union is the whole space unlike in T_2!)

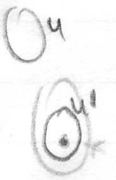
Rem $\mathbb{Q}$ does not satisfy the CNC but "$\Leftarrow$" of lemma is still true

CNC $\overset{(1),(2)}{\Longrightarrow}$ every union of conn. comps. is open and closed (which is false for $\mathbb{Q}$)

Rem:

Def: X locally connected at x

$\forall\, U \ni x$ neighborhood $\exists\, U' \subset U$ $U' \ni x$ ngh. connected



X locally connected if l.c. at x $\forall x \in X$.

Rem: X locally conn. (at x) $\Longrightarrow$ CNC (at x)

Th. X is locally connected $\Longleftrightarrow \forall\, U \subset X$ open

$\forall\, C_x \subset U$ comp of U (in rel. top.)

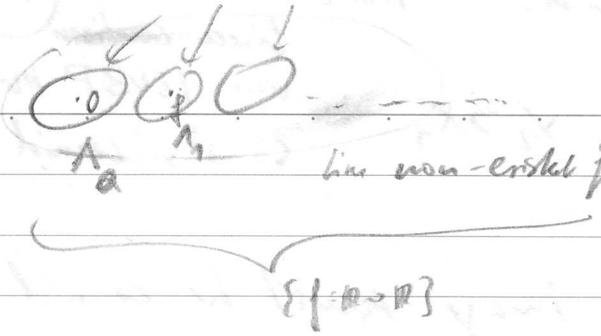
$C_x \subset X$ open

Ex. $\{f: \mathbb{R} \to \mathbb{R}\}$ with uniform top. $\mathcal{F}(\mathbb{R},\mathbb{R})_{uni}$

$f(x) = x \in \left\{ f : \lim_{x\to\infty} \frac{f(x)}{x} = 1 \right\} = A_1$ open & closed

$0 \in \left\{ \lim_{x\to\infty} \frac{f(x)}{x} = 0 \right\} = A_0$

disconnected! every behavior → gives a separation

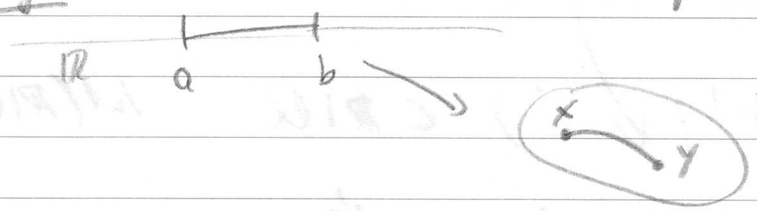


conn. comps. $[f]_\sim$

$f\sim g \iff f-g$ bounded

Path connectedness and path components

Def: path: $[a,b] \longrightarrow X$ from $x\in X$ to $y\in X$ is cont's $f(a)=x$, $f(b)=y$



X is path connected if $\forall x,y\in X$ $\exists$ path in X from x to y

write "$x \longleftrightarrow y$"

Rem path-conn $\Rightarrow$ conn.

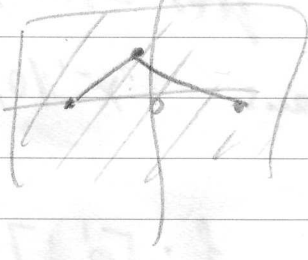
Ex. $V, \|.\|$ norm then unit ball $B=\{x\in V: \|x\|\le 1\}$

is convex $\forall x,y\in B$ $tx+(1-t)y\in B$

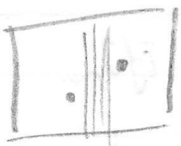
$t\longmapsto tx+(1-t)y$ cont's in t (in norm top.)

B is path conn'd

Ex. $\mathbb{R}^n\setminus\{0\}$ p.conn'd



Ex. I_0^2 is not path conn'd. but it's a linear continuum (LUBP now!) so it's conn'd



$a = (x_1, y_1)$ $\quad x_2 > x_1$

$b = (x_2, y_2)$

$f : [0,1] \to X$ cont's image must be conn'd

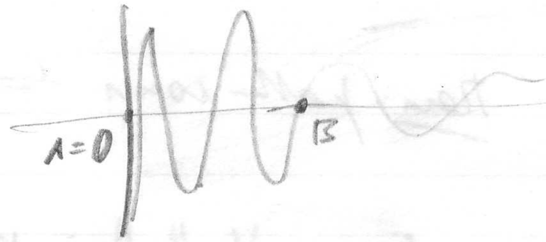
if image $\ni a, b \Rightarrow$ image $\supset [a,b] \supset \underbrace{(x_1, x_2) \times [0,1]}_{\text{unc.}}$

$\exists x_0 \in (x_1, x_2)$ $\quad f(\underbrace{Q \cap [0,1]}_{\text{cntble}}) \cap (\underbrace{\{x_0\} \times (0,1)}_{\text{open}}) = \emptyset$

$f^{-1}(\quad) \subset \mathbb{R} \setminus Q$ $\qquad \operatorname{Int}(\mathbb{R} \setminus Q) = \emptyset$ ↯

non-empty open as f is cont's

Ex. $A = \{x \times \sin(\frac{1}{x}) : 0 < x \leq \frac{1}{\pi}\} \subset \mathbb{R}^2$ $\quad$ topologist's sine curve



$\bar{A} = A \cup (\{0\} \times [-1,1])$

A conn'd $\Rightarrow \bar{A}$ conn'd

but $\bar{A}$ is not path conn'd

let there be a path $f : [0,1] \to \bar{A} \subset \mathbb{R}^2$

w.l.o.g. $f(t) = (x(t), y(t))$ with $f(0) = \underset{A}{(0,0)}$ $\quad f(1) = \underset{B}{(\frac{1}{\pi}, 0)}$

x continuous $\quad x^{-1}(0) \subset [0,1]$ closed

$\Rightarrow \exists \max x^{-1}(0) =: a'$

w.l.o.g. $f : [a', 1] \to \bar{A}$ $\quad x(t) > 0$ $\quad t > a' \Rightarrow \operatorname{img}(f|_{(a',1]}) \subset A$

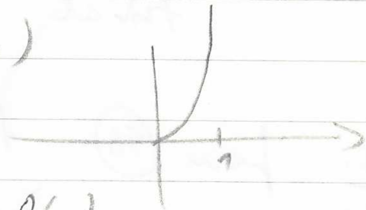
$x(a') = 0$, $x(1) = \frac{1}{\pi} \Rightarrow$ by IVT $\quad \forall x \in (0, \frac{1}{\pi}]$ $\quad \exists t$ $\quad x(t) = x \Rightarrow y(t) = \sin(\frac{1}{x})$

Image(f) $= A \cup \{0 \times y(a')\}$ $\qquad$ ↯

$[a', 1]$ (seq.) compact, f cont's $\Rightarrow$ Img(f) closed $\Rightarrow$ Img$(f) = \overline{A \cup \{0 \times y(a')\}} = \bar{A} \neq A \cup \{\ldots\}$

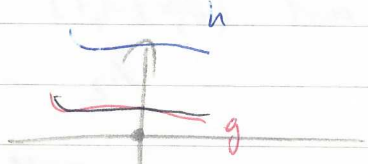
This means the closure of a path c. set is not path c.! Compare Th on (89)

Ex. $\mathcal{F}(\mathbb{R},\mathbb{R}_d) = \{f : \mathbb{R} \to \mathbb{R}_{discr}\}$ with product top.
is path conn. ($\Rightarrow$ conn.)

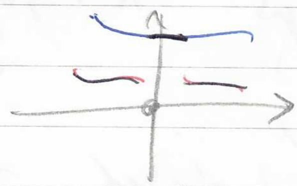


$g, h \in \mathcal{F}$

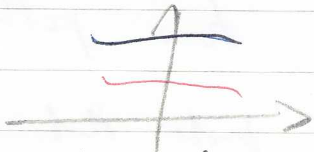
$$p \in [0,1] \quad f_p(x) = \begin{cases} h(x) & p = 1 \text{ or } p<1 \ |x| < \theta(p) \\ g(x) & p<1 \text{ and } |x| \geq \theta(p) \end{cases} \qquad \theta(x) = \tan\left(\frac{2x}{\pi}\right)$$



$f_0 = g$



$f_{0.3}$



$f_1 = h$

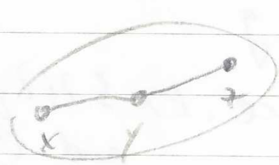
$p \mapsto f_p$ continuous $\quad f_0 = g \quad f_1 = h$

Rem: If $|S| = n < \infty$ $\quad \mathcal{F}(S, \mathbb{R}_{discr})_{prod} \cong \mathbb{R}^n_{discr}$ totally disconnected
$\mathcal{F}(\mathbb{Z}_+, \mathbb{R}_{discr})_{prod} = \mathbb{R}^\omega_{discr} = (\mathbb{R}^\omega, \text{dict order})$ also totally disconn.

Def: X top. space define eq. rel. by

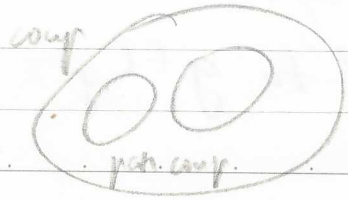
$x \sim y \quad \exists$ path in X from x to y

$[x]_\sim$ is x's path component



Th: path comp. of X are
disjoint path connected subspaces
whose union is X

each path-conn. subspace of X is
contained in exactly one path comp.



Ex. $\mathbb{Q} \subset \mathbb{R}$

all comp's trivial $\Rightarrow$ all path comp's are trivial

"totally path-disconnected"

Ex. from 96 topologist's sine curve is conn'd but not path c.

A

it has one comp, $\bar{A}$, and

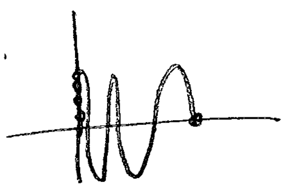
two path comps $\underset{A_1}{A \setminus \{0\times 0\}}$ and $\underset{A_2}{\{0\}\times[-1,1]}$

note that in A,

A_1 is open but not closed $\Rightarrow$ unlike comps., path comps. need not be closed

A_2 is closed but not open

Ex. cont'd take $A \setminus \{0\times 0\} \cup \{0\}\times([-1,1]\setminus\mathbb{Q})$



This is still connected (= one comp)

but has uncountably many path comp's !

Ex. Cons $\{f : \mathbb{R}\to\mathbb{R}\}$ with uniform topology

conn. comp of f are the fcts g with $|g-f|$ bounded (see Ex 3.76.2 p.160) and 95

This is path connected in fact it is convex (as cvs / $\mathbb{R}$)

$tf + (1-t)g$ gives a (straight) path from f to $g \in C_f$

Similarly you can do in box topology with

$C_f = \{g : \ |\{x : g(x) \neq f(x)\}| < \infty\}$

(is also convex)

Again this is a local version

Def: X is locally path connected

if $\forall U \ni x$ open $\exists U \supset U' \ni x$ U' path conn.

~~Lem~~ locally p.c. $\Rightarrow$ locally conn'd

Ex. top's sine curve is connected not path conn

and not locally connected $\Rightarrow$ not locally path-c.

Ex. $(0,1) \times [0,1)$ with dict order is

not conn. not locally connected

($\Rightarrow$ not path conn and not locally path conn.)

Ex. $[0,1) \times (0,1)$ with dict. order is

not conn. but locally conn.

not path conn. but locally path conn.

Ex. every set with the discrete top is

locally conn and lo. path conn since $U = \{x\} \ni x$. is open

(but, of course, totally disconnected and path-disconnected)

Th. X locally (p)c. $\Leftrightarrow$ $\forall U \subset X$ open

$\forall U'$ (p.)comp of U

$U' \subset X$ open

Th each p.comp of X lies (entirely) with a comp of X

If X is locally p.c. then comp's and path comp's are the same

Cor connected and locally path conn $\Rightarrow$ path conn.

Ex. $I_0^2 = [0,1]^2$ has LUBP (HW1) $\Rightarrow$ Linear continuum
Intermediate el. prop ICP $\Rightarrow$ connected
see Ph 3 p.160 Th. p.90 and locally conn.

path connected?
$(a,b) \longleftrightarrow (c,d)$ if $a=c$ 

but $(a,b) \not\longleftrightarrow (c,d)$ if $a \neq c$

assume $f: [0,1] \longrightarrow I_0^2$ $f(0) = (a,b) = a\times b$, $f(1) = (c,d) = c\times d$

$f([0,1]) \supset [a\times b, c\times d]$

$A = \mathbb{Q} \cap [0,1] \subset [0,1]$ dense, countable

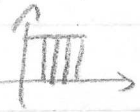
f cont $\Rightarrow$ $\overline{f(A)} \supset f(\bar{A}) = f([0,1]) \supset [a\times b, c\times d]$

$B := f(A) \cap (a\times b, c\times d)$ dense in $(a\times b, c\times d)$, countable

but $(a\times b, c\times d) \supset \bigcup_{a<x<b} \{x\} \times [0,1]$ $\Rightarrow \exists\, s \in (a,b)$
uncount. disjoint union

$B \cap (\{s\} \times [0,1]) = \emptyset$ $(a\times b, c\times d) \supset \{s\} \times [0,1] \supset \{s\} \times (0,1)$ open

$\bar{B} \not\supset (a\times b, c\times d)$ ↯

So path comps of I_0^2 are $\{s\} \times [0,1]$ 

§ 43 Complete metric spaces

Assume (X,d) metric space.

recall (proof of Borel-Lebesgue) $(X,d) \supset (x_n)$

$x_n \to x \implies \forall \varepsilon \; \exists N \; \forall m,n \geq N \; d(x_m, x_n) < \varepsilon$ (·)

Def: We say (x_n) is Cauchy-sequence if (x_n) satisfies (·)

So

Lemma (x_n) converges $\implies (x_n)$ Cauchy-seq.

Def: We say (X,d) is complete if every Cauchy sequence converges (i.e. converse of lemma holds)

Lemma Let (x_n) be a C-seq. If (x_n) has a conv. subseq., then (x_n) converges

Cor X complete $\iff$ every Cauchy seq. has a conv. subseq.

101a !

Cor. X compact $\implies$ X complete

top. prop. metric prop.

metrizable top. space compact $\implies$ complete with resp. to every metric inducing the top.

101a

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$$(0,1) \simeq \mathbb{R} \quad \text{homeo.}$$

Eucl. Eucl.

non-complete complete

so $\exists$ metric on $(0,1)$ giving Eucl. top.
w.r.t. which $(0,1)$ is complete

This completeness depends on metric (not only on metrizability)

Th. $\mathbb{R}^k$ complete. (with Eucl. metric)

Pf. let (x_n) be C-seq.
$\Rightarrow (x_n)$ bounded $\Rightarrow (x_n) \subset [-M, M]^k$ compact

$\Rightarrow (x_n)$ has conv subseq $\Rightarrow (x_n)$ converges ∎

For $\mathbb{R}^\omega$ recall the foll lemma.

Lemma let $X = \prod_{\alpha \in I} X_\alpha$ with product top, and
for $\alpha_0 \in I$ let $\pi_{\alpha_0} : X \to X_{\alpha_0}$
be the proj $(x_\alpha)_{\alpha \in I} \longmapsto x_{\alpha_0}$.
Then $\bar{x}_n \to \bar{x}$ in $X \Leftrightarrow \forall \alpha \in I.\ \pi_\alpha(\bar{x}_n) \to \pi_\alpha(\bar{x})$ in X_α

(i.e. convergence in the product top is pointwise convergence)

Th The product top on $\mathbb{R}^\omega$ has a metric w.r.t. which it is complete.

Pf. $d(\bar{x}, \bar{y}) = \sup_{i > 0} \left\{ \frac{\bar{d}(x_i - y_i)}{i} \right\}$ $\qquad \bar{d}(a,b) = \min(|a-b|, 1)$

d gives product top.

assume $(\bar{x}_n)$ C-seq. in $(X = \mathbb{R}^\omega, d)$

$\forall i$ $\pi_i(\bar{x}_n)$ in $\mathbb{R}$ is C-seq. bec.

$$|\pi_i(\bar{a}) - \pi_i(\bar{b})| \leq i\, d(\bar{a}, \bar{b}) \quad i \text{ fixed}$$

so $\pi_i(\bar{x}_n) \to a_i$ conv in $\mathbb{R}$ $\forall i$. Then

$$\bar{x}_n \to (a_i)_{i=1}^\infty \text{ in } X. \quad \blacksquare$$

Rem: (x_n) C-seq. w.r.t. d $\Leftrightarrow$ (x_n) C-seq w.r.t. $\bar{d}$ (X,d) complete $\Leftrightarrow$ $(X,\bar{d})$ complete

Ex. $\mathbb{Q}$, $(-1,1)$ with Eucl. metric not complete

cons $x_n \in \mathbb{Q}$ $x_n \to \sqrt{2}$ in $\mathbb{R}$ (for $\mathbb{Q}$)

$-1 + \frac{1}{n} \in (-1,1) \to -1$ (for $(-1,1)$)

Rem: $(-1,1) \overset{\text{homeo}}{\cong} \mathbb{R}$ not c. com. so $\exists \bar{d}$ on $(-1,1)$ giving Eucl. top. w.r.t. $(-1,1)$ complete $\to$ completeness depends on metric (not only on metrizability) $\Leftarrow$ 101 bottom

Th. Let (X,d) complete. $A \subset X$ closed $\Rightarrow$ $(A, d|_{A \times A})$ complete

Pf. $(x_n) \subset A$ Cauchy seq. $(x_n) \subset X$ cauchy $(x_n) \to x$ in X

A closed $\Rightarrow$ $x \in A$ $\Rightarrow$

$$(x_n) \to x \text{ in } A \quad \blacksquare$$

Rem. $\Leftarrow$ also true: $A \subset X$ complete $\Rightarrow$ A closed

$\mathbb{R}^J = \mathcal{F}(J, \mathbb{R})_{\text{prod}}$ is in general not metrizable, so completeness makes no sense

but $\mathbb{R}^J_{\text{uni}}$ is metrizable. Recall uniform metric

<u>Def:</u> Let (Y_α, d_α) metric spaces. Let

$\bar{d}_\alpha = \min(d_\alpha, 1)$

For $Y = \prod_{\alpha \in J} A_\alpha$ define the uniform metric

on Y by

$\bar{\rho}(\bar{x}, \bar{y}) := \sup_{\alpha \in J} \{ \bar{d}_\alpha (\pi_\alpha(\bar{x}), \pi_\alpha(\bar{y})) \}$

$\pi_\alpha : Y \to A_\alpha$

<u>Th.</u> (Y, d) complete $\Rightarrow$ $Y^J = \prod_{\alpha \in J} Y_\alpha$ $\quad$ $Y_\alpha = Y$

is complete with uniform metric. $\quad$ index the copy

<u>Pf.</u> Let $(f_n) \subset Y^J$ C-seq. w.r.t. $\bar{\rho}$

then $(\pi_\alpha(f_n)) \subset Y_\alpha$ C-seq. in $(Y_\alpha, \bar{d})$

$\Rightarrow$ C-seq. in (Y_α, d)

$\pi_\alpha(f_n) \to y_\alpha$ in Y_α

Let $f : \alpha \mapsto y_\alpha$

We claim $f_n \to f$ in $(Y^J, \bar{\rho})$

Given $\varepsilon > 0$ choose N with $\forall n, m \geq N$

$\bar{d}(f_n(\alpha), f_m(\alpha)) < \varepsilon/2 \quad \forall \alpha \in J \; (\Leftarrow \bar{\rho}(f_n, f_m) < \varepsilon/2)$

$\xrightarrow{m \to \infty}$ $\bar{d}(f_n(\alpha), f(\alpha)) \leq \varepsilon/2$

$\bar{d}$ cont's

this holds $\forall \alpha \in J$ $\forall n \geq N$

$$\bar{\rho}(f_n, f) = \sup_\alpha \bar{d}(f_n(\alpha), f(\alpha)) \leq \varepsilon/2 < \varepsilon$$

$\Rightarrow \forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \bar{\rho}(f_n, f) < \varepsilon \Rightarrow f_n \to f$ ☒

Def. Now assume X is top. space

$$C(X, Y) \subseteq \mathcal{F}(X, Y) = Y^X$$

is $\{ f \in Y^X : f \text{ continuous} \}$

Def. $f : X \to (Y, d)$ bounded if $f(X) \subset Y$ is a bounded set

diam$(f(X)) < \infty$

$B(X, Y) \subset Y^X$

$= \{ f : X \to Y \ \ f \text{ bounded} \}$

Th. (Y, d) metric space

$B(X, Y)$ and $C(X, Y)$ (X top. space)

are closed in $(Y^X, \bar{\rho})$ and thf. complete.

Ex. $C(\mathbb{R}, \mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R})_{uni}$ closed (but not discrete)

$C(\mathbb{R}, \mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R})_{box}$ (closed and) discrete ← HW

$C(\mathbb{R}, \mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R})_{pnt}$ is dense (and not closed)

$\cup$

$\mathbb{R}[z]$ dense (Lagrange interpolation)

Completion

Def: (X,d), $(Y,\tilde{d})$ metric spaces

we say $f: X \to Y$ is isometry if

$$\forall x, \tilde{x} \in X \quad d(x,\tilde{x}) = \tilde{d}(f(x), f(\tilde{x}))$$

Rem: f iso $\Rightarrow$ injective

so f is also called an "isometric embedding"

Th. (Existence of completion) (X,d) metric space $\exists (Y,\tilde{d})$ complete m.s.

$f: X \to Y$ iso embedding

Def: If (X,d) metric space, $(Y,\tilde{d})$ complete m.s. $f: X \to Y$ iso embedding,

call $\overline{f(X)} \subset Y$ the completion of X

Rem: completion is unique up to isometry

Constr:

$$\mathcal{U}(X^\omega) = \{(x_1, x_2, x_3, \dots) \subset X \text{ Cauchy seq.}\} \subset X^\omega$$

let $\sim$ be eq. rel. on $\mathcal{U}(X^\omega)$

$$(x_i) \sim (x_i') :\Leftrightarrow d(x_i, x_i') \to 0$$

Then $Y = \Gamma(X) := \mathcal{U}(X^\omega)/\sim$

$$\tilde{d}([(x_i)], [(x_i')]) = \lim_{i \to \infty} d(x_i, x_i')$$

$f: X \hookrightarrow Y$ is given by $x \mapsto [(x, x, x, \dots)]$

Ex. $X = \mathbb{Q}$ with E. metric $\Gamma(\mathbb{Q}) = \mathbb{R}$ constr. of real #s