Linear algebra and Its Applications

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(When it finally works!!!)

Book: *Linear Algebra* (4th Edition), by Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence

Grades(tentative):

Attendance	:	5%
Recit.	:	10%
Quiz, HW	:	5% each
Midterm	:	25%
Final	:	50%

⟨ Syllabus rules ⟩

0 Sets, Numbers, Maps, Fields, Rings

0.1 Set theory

$$\text{set} = \{\text{collection of objects}\}$$

Ex $A = \{0, 1, 2\}$, $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$

some notations:

$$
\begin{array}{rcl}
x \in A & : & \text{element} \\
x \notin A & : & \text{not element} \\
A \subset B & : & A \text{ contained in } B\text{; when } x \in A\,,\ \text{then } x \in B \\
A \subsetneq B & : & A \subset B \textit{ and } A \neq B \\
\varnothing & : & \text{empty set(no element)}
\end{array}
$$

<u>Ex</u>

1. $x = 1,\ A = \{1, 2, 3\}$

   $\Rightarrow\ x \in A$

2. $x = 4\ \Rightarrow\ x \notin A$

<u>Ex</u> $1 \in \{1, 2, 3\}$

$$\{1\} \subset \{1, 2, 3\},\ \{1, 2, 3\} \subset \{1, 2, 3\},\ \{1, 3\} \subset \{1, 2, 3\}$$

$$\{1\} \subsetneq \{1, 2, 3\}\ \ \underline{\textit{not}}\ \{1, 2, 3\} \subsetneq \{1, 2, 3\}$$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

$$
\begin{array}{rcll}
A \cap B & = & \{x\ :\ x \in A \text{ and } x \in B\} & : \text{ intersection} \\
A \cup B & = & \{x\ :\ x \in A \text{ or } x \in B\} & : \text{ union} \\
A \setminus B & = & \{x\ :\ x \in A \text{ and not } x \in B\} & : \text{ set difference}
\end{array}
$$

What kind of numbers?

$\mathbb{N}$	natural numbers (0 incl.) $0, 1, 2, 3, \cdots$
$\mathbb{Z}$	integers $\cdots - 3, -2, -1, 0, 1, 2, 3, \cdots$
$\mathbb{Q}$	rationals $\frac{m}{n}$ $(m,\, n \in \mathbb{Z}, n \neq 0)$
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{H}$	quaternions (Hamilton numbers; later)

$\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$ <u>positive natural numbers</u>

if $A \subset X$, $X \textit{fixed}$, then $X \setminus A =: \overline{A}$ is called *complement* of A in X
if $A \cap B = \varnothing \quad \Leftrightarrow$: A,B *disjoint*
indexed $\cap$ and $\cup$

$\{\, A_i\ :\ i \in I \,\}$ sets
$$\bigcap_{i \in I} A_i\ :=\ \{\, x\ :\ x \in A_i \text{ for all } i \in I\ \}$$
$$\bigcup_{i \in I} A_i\ :=\ \{\, x\ :\ x \in A_i \text{ for at least one } i \in I\ \}$$

Ex. when $I = \mathbb{Q}$ and $A_i = \{ x \in \mathbb{Q} : x \geq i \}$ then $\bigcup_{i \in I} A_i = \mathbb{Q}$ and $\bigcap_{i \in I} A_i = \varnothing$ and

product set $A \times B := \{ (x, y) : x \in A, \ y \in B \}$

$\underbrace{A \times A \times \cdots \times A}_{n \text{ times}} =: A^n$

example: $\mathbb{R}^2 := \{ (x, y) : x, y \text{ real} \}$ coordinate plane

$\mathbb{R}^3 := \{ (x, y, z) : x, y, z \text{ real} \}$ (3-dimensional) Euclidean space

cardinality # of elements in the set (or ∞)

$|\varnothing| = 0$, $|\{1\}| = 1$, $|\mathbb{Z}| = \infty$

equivalence relation on a set X: $S \subset X \times X$ with 3 properties:

$\forall x \in X$:	$(x, x) \in S$	(reflexivity)
$\forall x, y \in X$:	if $(x, y) \in S$, then $(y, x) \in S$	(symmetry)
$\forall x, y, z \in X$:	if $(x, y), (y, z) \in S$, then $(x, z) \in S$	(transitivity)

We write $x \sim_S y$, or simply $x \sim y$, "x is *equivalent* to y", for $(x, y) \in S$.

Ex $S = \{(x, x) : x \in X\}$, $S = X \times X$ trivial equivalences

Ex $x, y \in X = \mathbb{Z}$, $\quad x \sim_n y$ if $n \mid (x - y)$ $(n \in \mathbb{N}, n > 0)$

0.2 Real numbers

Arithmetic

$$\frac{m \cdot k}{n \cdot k} = \frac{m}{n} \text{ reduction } (k \in \mathbb{Z}, k \neq 0) , \ \frac{m}{n} \pm \frac{p}{q} = \frac{mq \pm pn}{nq}$$

$$\frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq} , \ \frac{m}{n} / \frac{p}{q} = \frac{mq}{np} (n, p, q \neq 0)$$

Rational numbers are "dense(조밀)". But they are not all. E.g., $x = \sqrt{2} \notin \mathbb{Q}$; x is irrational

more generally if z is integer and $x = \sqrt{z}$ is not integer, then $\sqrt{z}$ is irrational.

0.2.1 Algebra of Real Numbers

Addition & multiplication

$$\begin{cases} \text{commutativity(교환법칙)} & a + b = b + a, \quad ab = ba \\ \text{associativity(결합법칙)} & (a + b) + c = a + (b + c), \quad a(bc) = (ab)c \end{cases}$$

but not subtraction & division

$$\begin{array}{ll} 3-5 \neq 5-3, & (3-1)-2 \neq 3-(1-2) \\ 3 \div 5 \neq 5 \div 3, & (3 \div 2) \div 5 \neq 3 \div (2 \div 5) \end{array}.$$

$$\begin{array}{ll} \text{implicit parentheses:} & a-b-c=(a-b)-c \\ \text{from left to right} & a \div b \div c = (a \div b) \div c \end{array}.$$

Order algebraic of operations

1. multiplication & division precede addition & subtraction

$$a+b\cdot c = a+(b\cdot c)\ [\neq (a+b)\cdot c]$$

2. evaluate innermost parentheses 1st

Ex $\quad 2\overbrace{(6+3\underbrace{\overbrace{(1+4)}}_{inner()})}^{outer()} = 2(6+3\cdot 5)$

$$= 2(6+15)$$

$$= 2\cdot 21 = 42$$

Distributive Property

$$(b+c)a = a(b+c) = ab+ac$$

$\longleftarrow$simplify

$\longrightarrow$expand

Examples of use:

$$(a+b)(c+d) = ac+ad+bc+bd$$

$$(a+b)^2 = a^2+2ab+b^2$$

Additive inverses & subtraction

additive inverse of $a \in \mathbb{R}\ -a\ :\ a+(-a)=0$

$$a-b\ =\ a+(-b)$$

Rules of additive inverse

$$
\begin{aligned}
-(-a) &= a \\
-(a+b) &= -a-b \\
(-a)(-b) &= ab \\
(-a)b = a(-b) &= -ab \\
(a-b)c &= ac-bc
\end{aligned}
$$

<u>multiplicative inverses & division</u>

multiplicative inverse of $b \in \mathbb{R},\ b \neq 0\ \frac{1}{b}(= b^{-1})\ :\ b \cdot \frac{1}{b} = 1$

$$\frac{a}{b} = a \cdot \frac{1}{b} = a \cdot b^{-1}$$

see rules for rational numbers

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd},\ \frac{1}{\frac{a}{b}} = \frac{b}{a},\ \frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b},\ \frac{-a}{-b} = \frac{a}{b}\ (\text{in particular } \frac{1}{-b} = -\frac{1}{b})$$

0.2.2 <u>Inequalities</u>

$a < b$	a less than b	(a left of b on real line)
$a > b$	a greater than b	
$a \leq b$	a less/smaller than or equal to b	
$a \geq b$	a greater/bigger than or equal to b	
$a < b,\ b < c$	$\Rightarrow\ a < c$	transitive

addition

$$a < b,\ c < d\ \Rightarrow\ a + c < b + d$$

multiplication

$$
\begin{aligned}
a < b \quad c > 0 \ &\Rightarrow\ ac < bc \\
c < 0 \ &\Rightarrow\ ac > bc
\end{aligned}
$$

$\Rightarrow$ (additive inverse) $a < b\ \Rightarrow\ -a > -b$
multiplicative inverse & inequalities

$$ab \neq 0, \quad a < b \quad \begin{array}{l} \frac{1}{a} < \frac{1}{b} \text{ if } ab < 0 \\ \frac{1}{a} > \frac{1}{b} \text{ if } ab > 0 \end{array}$$

<u>Ex</u> $ab > 0$

1. $a = 2,\ b = 3\quad \frac{1}{3} < \frac{1}{2}$
2. $a = -3,\ b = -2\quad -\frac{1}{2} < -\frac{1}{3}$

$ab < 0$ 3. $a = -2,\ b = 3 \quad -\frac{1}{2} < 0 < \frac{1}{3}$

Intervals

Let $a, b \in \mathbb{R},\ a \leq b$

$$\begin{array}{rcll} \text{set} & = & \{\text{objects with some property}\} & \\ (a,b) & = & \{x \in \mathbb{R}\ :\ a < x < b\} & \text{open interval} \\ (a,b] & = & \{x \in \mathbb{R}\ :\ a < x \leq b\} & \text{left-open intervals} \\ [a,b) & = & \{x \in \mathbb{R}\ :\ a \leq x < b\} & \text{right-open intervals} \\ [a,b] & = & \{x \in \mathbb{R}\ :\ a \leq x \leq b\} & \text{closed interval} \end{array}$$

Ex

1. $A = [0,2),\ B = (1,3)$

   $A \cap B = (1,2),\ A \cup B = [0,3)$

   $A \setminus B = [0,1],\ B \setminus A = [2,3)$

2. $A = [0,1],\ B = [1,2)$

   $A \cap B = 1$

3. $A = [0,1],\ B = (1,2)$

   $A \cap B = \varnothing$ (A,B disjoint)

∞ :infinity, $\Rightarrow \infty > a\ \forall a \in \mathbb{R}$

$-\infty$:negative infinity, $\Rightarrow -\infty < a\ \forall a \in \mathbb{R}$

$$\begin{array}{rcl} (a,\infty) & = & \{x \in \mathbb{R}\ :\ a < x\} \\ [a,\infty) & = & \{x \in \mathbb{R}\ :\ a \leq x\} \\ (-\infty,a) & = & \{x \in \mathbb{R}\ :\ x < a\} \\ (-\infty,a] & = & \{x \in \mathbb{R}\ :\ x \leq a\} \end{array}$$

(Note: '$[-\infty,$' or '$,\infty]$' make no sense, since $\infty,\ -\infty \notin \mathbb{R}$)

Absolute value

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Ex $|\frac{3}{2}| = \frac{3}{2},\ |-2| = 2,\ |0| = 0$

Ex $\{x \in \mathbb{R}\ :\ |x| < 2\} = (-2,2)$

0.3 Functions and Their Graphs

0.3.1 Functions

Domain(정의역) and target(공역)

function f from A and B associates to each $a \in A$ (*argument*) an element $f(a) \in B$ (*value*)

$$f : A \longrightarrow B$$

$$\begin{array}{lll} A & : & \text{domain of } f \quad \mathit{domain}(f) \\ B & : & \text{target of } f \quad \mathit{target}(f) \supseteq \mathit{range}(f) \end{array}$$

Ex $f(x) = x^2, x \in \mathbb{R}$

Then $f(3) = 3^2 = 9$

$f(-\frac{1}{2}) = (-\frac{1}{2})^2 = \frac{1}{4}$

Ex f does not need to be defined by a single expression.

$$g(x) = \begin{cases} 3x & \text{if } x < 0 \\ \sqrt{2} & \text{if } x = 0 \\ x^2 + 7 & \text{if } x > 0 \end{cases}$$

These conditions must be disjoint!

Domain=$\{x \in \mathbb{R} \ : \ \text{some condition for } x\}$

$$\begin{array}{lll} g(-2) & = & 3 \cdot (-2) = -6 \\ g(0) & = & \sqrt{2} \\ g(1) & = & 1^2 + 7 = 8 \end{array}$$

Equality of functions

Definition : Two functions f, g are equal if

$$\mathit{domain}(f) = \mathit{domain}(g),\ \mathit{target}(f) = \mathit{target}(g) \text{(not in book!)},$$

and for all $x \in \mathit{domain}(f)$ we have $f(x) = g(x)$.
Ex

$$f : \mathbb{R} \longrightarrow \mathbb{R}\,,\quad f(x) = 3x$$

$$g:\mathbb{R}\longrightarrow\mathbb{R}\,,\quad g(t)=3t$$

$$f=g\ \ \text{(how you call the variable in definition does not matter)}$$

<u>Ex</u>

$$f:\mathbb{R}\longrightarrow\mathbb{R}\,,\quad f(x)=3x$$

$$g:\mathbb{R}\longrightarrow\mathbb{R}\,,\quad g(x)=\sqrt{(3x)^2}$$

$$f=g\text{(how you call the variable in definition does not matter)}$$

$$f\neq g$$

because $x=-1\ \ g(x)=\sqrt{9}=3,\ f(x)=3\cdot(-1)=-3$

<u>Ex</u>

$$f:\mathbb{R}\longrightarrow\mathbb{R}\,,\quad f(x)=3x$$

$$g:\{x\in\mathbb{R}\ :\ x>0\}\longrightarrow\mathbb{R}\,,\quad g(x)=3x$$

$$f=g\ \ \text{(how you call the variable in definition does not matter)}$$

$$f\neq g$$

because $domain(f) \neq domain(g)$

<u>Definition</u> : Assume $A \subseteq B \subseteq \mathbb{R}$ and $f : B \longrightarrow \mathbb{R}$ function

Then the function $g : A \longrightarrow \mathbb{R}$ given by $g(x) = f(x)$ for all $x \in A$

is called the restriction of f to A and written

$$g = f\Big|_A .$$

[(in the above ex. $g = f\Big|_{\{x\in\mathbb{R}\,:\,x>0\}}$)]

<u>Ex</u> $A = 1, 2\;\; f(x) = x^2,\;\; g(x) = 3x - 2$

$$f(1) = g(1) = 1,\; f(2) = g(2) = 4\; \rightarrow\; f = g$$

so functions can be equal even if given by very different formulas.

<u>Domain</u>

<u>Convention</u>(협약): If no domain is given, we assume that the domain is the maximal subset of $\mathbb{R}$ where definition makes sense.

<u>Ex</u> $f(x) = \dfrac{1}{3x-4}$

$$\rightarrow\; domain(f) = \{x \in \mathbb{R}\;:\; 3x - 4 \neq 0\} = \mathbb{R} \setminus \{\frac{4}{3}\}$$

<u>Ex</u> $f(x) = \sqrt{3x-4}$

$$\rightarrow\; domain(f) = \{x \in \mathbb{R}\;:\; 3x - 4 \geq 0\} = [\frac{4}{3}, \infty)$$

<u>Ex</u> $f(x) = x + 1,\; g(x) = \dfrac{(x+1)(x+2)}{(x+2)}$

$$domain(f) = \mathbb{R},\; domain(g) = \{x \in \mathbb{R}\;:\; x + 2 \neq 0\} = \mathbb{R} \setminus \{-2\}$$

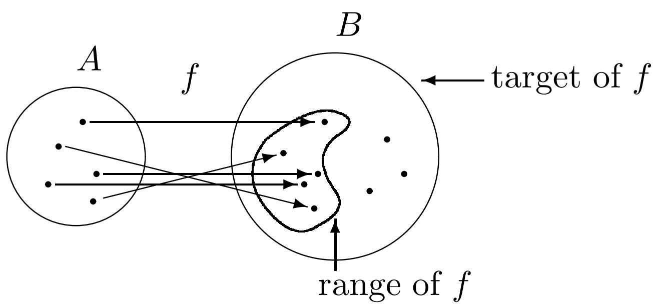
g is not defined in -2!

<u>Functions via tables</u>

x	$f(x)$
0.1	1.01
0.2	1.04
0.3	1.25
0.4	1.39

<u>Range(치역) of a function</u>

<u>Definition</u> : Range of a function $f : A \longrightarrow B$ is all $b \in B$ for which these is at least one $a \in A$ with $f(a) = b$



<u>Ex</u> $f : \mathbb{R} \longrightarrow \mathbb{R} \;\; f(x) = |x|$

domain$= \mathbb{R}$, target $= \mathbb{R}$, range $= \{x \in \mathbb{R} \; : \; x \geq 0\}$

<u>Ex</u> $f=$

$$\textit{domain}\left\{\begin{array}{c|c} x & f(x) \\ \hline \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{array}\right\}\textit{range}$$

when f is given by a table

<u>Ex</u> $f = 3x + 1 \;\; \text{domain}(f) = [-2, 5]$

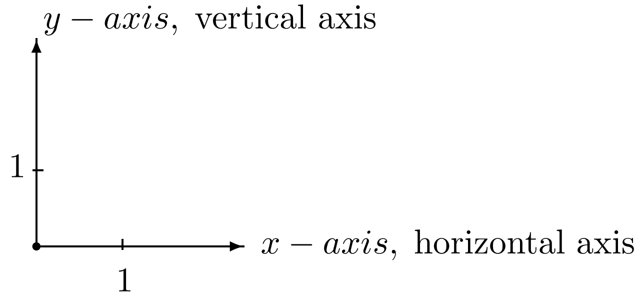
Is $19 \in \text{range}(f)$?

$19 = f(x) = 3x + 1, \; -2 \leq x \leq 5$

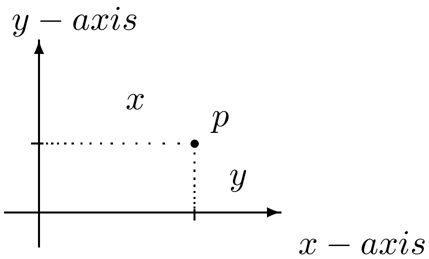
$\rightarrow \; x = \frac{18}{3} = 6 \; \notin [-2, 5]$

so no x exists $\rightarrow \; 19 \notin \text{range}(f)$

0.4 Coordinate Plane and Graphs



coordinate plane



$$p = (x, y) \;\; \text{rectangular(Cartesian) coordinates of } p$$

Graph of a function

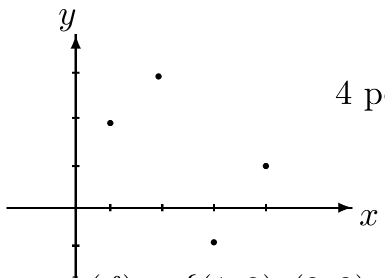
$f : A \longrightarrow B \;\; A, B \subset \mathbb{R}$

The graph of f consists of all points $(x, f(x))$ for $x \in A$

$$\text{graph}(f) \; := \; \{\, (x, f(x)) \; : \; x \in A \,\} \; \subset \; A \times B\,.$$

Ex $A = \{1, 2, 3, 4\}$

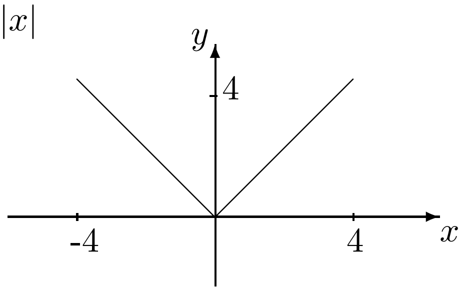
x	$f(x)$
1	2
2	3
3	-1
4	1



4 points

$$graph(f) = \{(1, 2), (2, 3), (3, -1), (4, 1)\}$$

Ex $A = [-4, 4] \;\; f(x) = |x|$



Even and Odd functions

$f(x) = x^2$ Then reflecting graph of f with respect to $y - axis$ gives graph itself. Because $f(x) = x^2 = (-x)^2 = f(-x)$

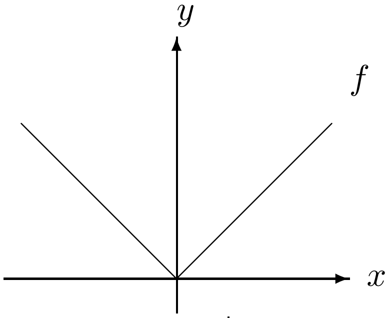
Definition: f is called even(우함수, 짝함수) function if

$$f(x) = f(-x) \text{ for every } x \in domain(f)$$

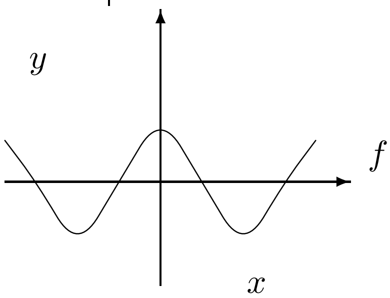
(Note:This means in particular that $x \in domain(f) \Rightarrow -x \in domain(f)$. For example $f(x) = x^2$ for domain(f) $= [-1, 1)$ is not even!)

other examples $f(x) = |x|$ ($domain(f) = \mathbb{R}$)

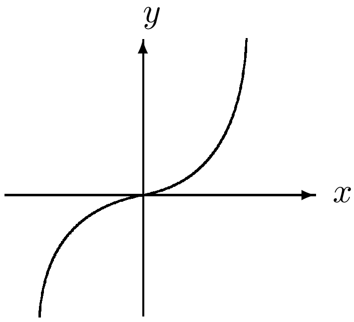
Because $|-x| = |x|$



$f(x) = \cos x$



$f(x) = x^3$



Graph(f) has rotational symmetry by 180° around origin

$\Leftrightarrow$ graph(f) is mapped to itself
when we mirror with respect to both x- and $y - axis$

given f what function is graph f after mirroring with respect to both x- and y-axis

$$f(x) \xrightarrow[\text{mirroring}]{\text{y-axis}} f(-x) \xrightarrow[\text{mirroring}]{\text{x-axis}} -f(-x)$$

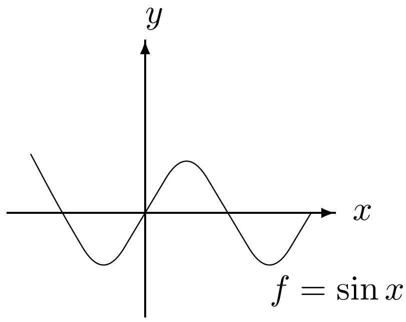
Graph(f) has rotational symmetry by 180° around origin

$\Leftrightarrow f(x) = -f(-x)$

These are odd functions.
Definition $f : A \to B$ odd(기함수, 홀함수) if $\forall x \in A$ ($-x \in A$ and)

$$f(x) = -f(-x)$$

Ex $f(x) = x^n$ n odd
in particular $f(x) = x$



0.5 Composition of functions

Ex $h(x) = \sqrt{x+3}$ is calculated in 2 steps:

1. add 3
2. take the root

These correspond to two separate functions $g(x) = x + 3$ and $f(y) = \sqrt{y}$. So $h(x) = f(g(x))$.

We need this often, so we make definition.
Definition The composition of f and g

is defined by $(f \circ g)$=$f(g(x))$.

This is defined when $x \in domain(g)$ and $g(x) \in domain(f)$.

Thus $domain(f \circ g) \subset \{x \in domain(g) \ : \ g(x) \in domain(f)\}$.

(and "=" if we do not specify domain($f \circ g$))per convention!
Ex Let $f(y) = \dfrac{1}{y-4}$, $g(x) = x^2$

1. evaluate $(f \circ g)(3)$
   sol: $(f \circ g)(3) = f(g(3)) = f(3^2) = f(9) = \frac{1}{9-4} = \frac{1}{5}$
2. find a formula for $(f \circ g)$
   sol: $(f \circ g)(x) = f(g(x)) = f(x^2) = \dfrac{1}{x^2 - 4}$
3. determine domain$(f \circ g)$
   sol: $domain(f \circ g) = \{x \in \underbrace{domain(g)}_{\mathbb{R}} \ : \ g(x) \in \underbrace{domain(f)}_{\{y \in \mathbb{R} \,:\, y \neq 4\}}\}$
   $= \{x \in \mathbb{R} \ : \ x^2 \neq 4\} = \mathbb{R} \setminus \{-2, 2\}$

Composition is associative : $f \circ (g \circ h) = (f \circ g) \circ h =: \ f \circ g \circ h$

Composition is not commutate : $f \circ g \neq g \circ f$ in general
e.g. Let $f(x) = x^2, \ g(x) = x + 1$

$f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1, \ g(f(x)) = g(x^2) = x^2 + 1$

$f(g(x)) \neq g(f(x))$
Identity Function

$$I : A \to A, \quad I(x) = x, \quad I = I_A$$

when $f : A \to A$, then $I \circ f = f = f \circ I$
(I is the identity for the operation of composition)

Sometimes one can write $h = f \circ g$ for f, g simpler than h.
Such decomposition $h = f \circ g$ is not unique.

$$h = \sqrt{\frac{x^2+3}{x^2+1}} \qquad \begin{array}{l} h = f \cdot g \quad f(y) = \sqrt{y} \ \ g(x) = \dfrac{x^2+3}{x^2+1} \\ h = \tilde{f} \circ \tilde{g} \quad \tilde{f}(y) = \sqrt{\dfrac{y+3}{y+1}} \ \ \tilde{g}(x) = x^2 \end{array}$$

0.6 Inverse functions

Ex. $f(x) = 3x$ find x with $f(x) = 6$
sol. $3x = 6 \ \Rightarrow \ x = 2$
$f(x) = y$
solve for x sol. $3x = y \ \Rightarrow \ x = \frac{y}{3}$
(exactly one x exists)
$x =: f^{-1}(y) = \frac{y}{3}$
This can be defined if for given y there is exactly one x.
This is of course not always the case.
$f(x) = x^2 + 1$ $[dom(f) = \mathbb{R}]$
$x^2 + 1 = y$
What is $f^{-1}(y)$?

$$x = \pm\sqrt{y-1}, \qquad \text{thus} \quad \left\{ \begin{array}{ll} y > 1 & \text{2 values for } x \\ y = 1 & \text{1 value for } x \\ y < 1 & \text{0 values for } x \end{array} \right\} \tag{1}$$

We must change domain(f) so that only one x occurs.
Definition: Let $f : A \to B$ and $y \in B$, $B' \subset B$
We define $f^{-1}(B') = \{x \in A \ : \ f(x) \in B'\}$
$f^{-1}(y) \subseteq A$ (set!) as $f^{-1}(y) = \{x \in A \ : \ f(x) = y\} = f^{-1}(\{y\})$
f is injective if for all $y \in B$, $\ | f^{-1}(y) | \leq 1 \qquad \Leftrightarrow$
$\forall x, y \in A \ : \ f(x) = f(y) \Rightarrow x = y$
f is surjective if for all $y \in B$, $\ | f^{-1}(y) | \geq 1 \qquad \Leftrightarrow \qquad range(f) = B$
($| \ |$ means number of elements)

f is bijective(book: one-to-one) if f is surjective and injective
$\Leftrightarrow \mid f^{-1}(y) \mid = 1$, $f^{-1}(y) = \{x\}$

Definition Then we can define an inverse function f^{-1}
$f^{-1}(y) := x$ for the x with $f(x) = y$.
(when f is bijective, x is unique.)

Domain and Range of inverse function:
domain(f^{-1})=range(f)
range(f^{-1})=domain(f)

The Composition of a function and its inverse
$f^{-1}(f(x)) = x \quad \forall x \in \text{domain}(f)$
$f(f^{-1}(y)) = y \quad \forall y \in \text{range}(f) = \text{domain}(f^{-1})$
$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x = Id(x)$
$f^{-1} \circ f = Id_{domain(f)} \qquad f \circ f^{-1} = Id_{range(f)}$
$\uparrow$ You can use to check formula for f^{-1}

Ex $f(x) = \frac{9}{5}x + 32 \quad (x^\circ \mathbf{C} = f(x)^\circ \mathbf{F})$
$f^{-1}(y) = \dfrac{5(y-32)}{9}$
check $(f^{-1} \circ f)(x) = f^{-1}(\frac{9}{5}x + 32) = \dfrac{5((\frac{9}{5}x + 32) - 32)}{9} = \dfrac{5 \cdot \frac{9}{5}x}{9} = x$

Comments about notation:
variable name does not matter.
1) $f^{-1}(x) = 5x - 37\sqrt{x}$
$f^{-1}(y) = 5y - 37\sqrt{y}$ are equivalent statements
I choose y for argument of f^{-1} and x for argument of f to indicate that x and y are possibly in two different sets $x \in \text{domain}(f)$, $y \in \text{range}(f)$.
2) $f^{-1}(y) \neq f(y)^{-1} = \dfrac{1}{f(y)}$
ex. $f(x) = x^2 \ (x \geq 0)$, $\quad f^{-1}(y) = \sqrt{y}$

$$f(y)^{-1} = \frac{1}{f(y)} = \frac{1}{y^2}$$

Thus if "f^{-1}" is written be careful how it is meant!

0.7 Group, Field, Ring

Definition A group $(G, +)$, $(G, \cdot)$ (additive/ multiplicative notation)
G set and $\cdot : G \times G \to G$ map "operation"
$\cdot(g_1, g_2) =: g_1 \cdot g_2 = g_1 g_2$
with the following properties

1) $\exists 1 \in G$ 1-element or neutral element, identity (book)

$$1 \cdot g = g \cdot 1 = g \quad \forall g \in G$$

2) $\forall g \in G \ \exists g' \in G : g \cdot g' = g' \cdot g = 1, \qquad g' = g^{-1}$ inverse of g

3) $g_1(g_2g_3) = (g_1g_2)g_3 \; \forall g_1, g_2, g_3 \in G$ (associativity)

Definition If additionally

4) $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G$, then G is an Abelian (commutative) group

example $(\mathbb{Z}, +)$ is Abelian group with neutral element 0

Def $(F, +, \cdot)$ is a field if $\begin{bmatrix} + & \text{addition} \\ \cdot & \text{multiplication} \end{bmatrix}$

$\exists 0, 1 \in F$ s.t.

1) $(F, +)$ is an Abelian group with neural element 0

2) $(F \setminus \{0\}, \cdot)$ is an Abelian group with neural element 1

3) $+, \cdot$ are distributive: $(a + b) \cdot c = ac + bc \; \forall a, b, c \in F$

if in $(F \setminus \{0\}, \cdot)$

	exists inverse	no inverse $(F \setminus \{0\}, \cdot)$ monoid	with or w/t 1
commutative	F is field	F is commutative ring	with or w/t 1
noncommutative	F is skew-field	F is noncommutative ring	with or w/t 1

Rem additive inverse of b in $(F, +)$ is written as $-b$ and $a - b = a + (-b)$ for mult. inverse we write $b^{-1} = \frac{1}{b}$, and $\frac{a}{b} = a \cdot b^{-1} = b^{-1} \cdot a$ (Note: in a skew-field, $\frac{a}{b}$ makes no sense!)

Ex $(\mathbb{C}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$ are fields
$(\mathbb{Z}, +, \cdot)$? but $\cdot$ has no inverse e.g. $2 \in \mathbb{Z} \; \nexists g \in \mathbb{Z} : \; 2 \cdot g = 1$
$\Longrightarrow$ comm. ring with 1, but no field
$(2\mathbb{Z}, +, \cdot)$ now $\cdot$ has no inverse and no identity $[\nexists e \in 2\mathbb{Z} : \; e \cdot k = k \; \forall k \in 2\mathbb{Z}]$
$\Longrightarrow$ comm. ring without 1

Th. (cancellation laws) F field, $a, b, c \in F$ arbitrary

1) $a + b = c + b \Rightarrow a = c$

2) $b \neq 0$, $a \cdot b = c \cdot b \Rightarrow a = c$

Corollary The identity elements $0, 1 \in F$ are unique.

Th. In any field F,

1) $a \cdot 0 = 0 \cdot a = 0$

2) $-(a \cdot b) = (-a) \cdot b = a \cdot (-b)$

Corollary The additive identity 0 in F has no multiplicative inverse.

0.7.1 characteristic

Def F field. Define the characteristic $\mathrm{char}(F) \in \mathbb{N}$ by

$$\mathrm{char}(F) := \begin{cases} \min\{\, n \in \mathbb{N}_+ \;:\; \underbrace{1+1+\cdots+1}_{n \text{ times}} = 0 \,\} & \text{if such an } n \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Ex For $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, $\mathrm{char}(F) = 0$ (no n)

Let $n > 1$. Consider
$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{$ congruence classes mod$n$ $\} = \{0, 1, 2, \ldots, n-1\}$.
$(\mathbb{Z}_n, +, 0)$ additive Abelian group
$(\mathbb{Z}_n, \cdot, 1)$ everything ok except inverse

Lemma $\exists$ mult. inverse $\iff$ n prime

So $\mathbb{Z}_n$ is a field for n prime, but only a ring for other n (cyclic field/ring)
For n prime, $\mathrm{char}(\mathbb{Z}_n) = n$.

Rem Only 0 and primes can be characteristic of a field.

Rem If $\mathrm{char}(F) = 1$, then $1 = 0 \Rightarrow F = \{0\}$, not interesting.

0.8 Complex numbers

0.8.1 arithmetic, norm, conjugate

$\mathbb{C}$ complex numbers
$z \in \mathbb{C}$ is of the form $z = a+bi$, $a, b \in \mathbb{R}$, $a = \Re e z$ real part
$b = \Im m z$ imaginary part
$i = \sqrt{-1}$ (imaginary unit)

$z = a + bi$
$w = c + di$

$z + w = (a+c) + (b+d)i \qquad -z = -a - bi$
$zw = z \cdot w = (ac - bd) + (ad + bc)i$
if $z \in \mathbb{R}$ (i.e., $b = 0$), $zw = (zc) + (zd)i$

$\overline{z} = a - bi$ (complex) conjugate

$z \cdot \overline{z} = a^2 + b^2 = |z|^2 \qquad |z| = \sqrt{a^2 + b^2} \in \mathbb{R}$ norm
$[\quad |z| = 0 \iff a = b = 0 \iff z = 0 \quad]$ abs. value

$$\Longrightarrow \text{for } z \neq 0,\ z^{-1} = \frac{1}{|z|^2}\overline{z} = \left(\frac{a}{a^2+b^2}\right) - \left(\frac{b}{a^2+b^2}\right) i$$

<u>Th</u> $\mathbb{C}$ forms a field.

<u>Th</u> The conjugation has the following properties.

a) $\overline{\overline{z}} = z$

b) $\overline{z+w} = \overline{z} + \overline{w}$

c) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

d) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}} \quad (w \neq 0)$

e) $z \in \mathbb{R} \iff z = \overline{z}$

f) $z + \overline{z} = 2\Re e z, \quad z - \overline{z} = 2i\Im m z$

<u>Th</u> The norm has the following properties.

a) $|z| = |\overline{z}|$

b) $|z| \geq |\Re e z| \quad |z| \geq |\Im m z|$

c) $|zw| = |z| \cdot |w|$

d) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \quad (w \neq 0)$

e) $|z| - |w| \ \leq \ |z+w| \ \leq \ |z| + |w|$

<u>Pf</u> a), b) exercise

c) $|zw|^2 \ = \ zw\overline{zw} \ = \ z\overline{z}w\overline{w} = |z|^2|w|^2 = (|z||w|)^2$

d) $\left|\frac{1}{w}\right| = \frac{1}{|w|}$ because $|w| \cdot \left|\frac{1}{w}\right| = \left|w \cdot \frac{1}{w}\right| = |1| = 1$

so use then c)

e)

$$\begin{aligned} |z+w|^2 = (z+w)(\overline{z}+\overline{w}) & \overset{\substack{\text{f) previous theorem} \\ \downarrow}}{=} z\overline{z} + 2\Re e(z\overline{w}) + w\overline{w} \\ & \overset{b)}{\leq} |z|^2 + 2|z\overline{w}| + |w|^2 \\ & \overset{a,c)}{=} |z|^2 + 2|z||w| + |w|^2 \ = \ (|z|+|w|)^2 \end{aligned}$$

first inequality a consequence of second:

$|-z| \overset{c)}{=} |-1| \cdot |z| = |z|$

$|z| = |(z+w) + (-w)| \ \leq \ |z+w| + |-w| \ = \ |z+w| + |w|$

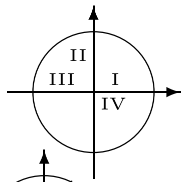
bring $|w|$ on other side $\Longrightarrow \square$

0.8.2 Trigonometric Functions (삼각함수)

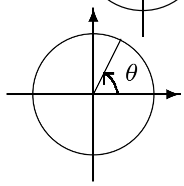
The unit circle unit circle: circle with center $(0,0)$, radius$=1$
equation: $x^2+y^2=1$

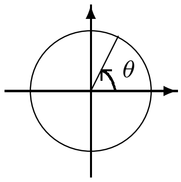
Angles in the unit circle

I :	positive horizontal axis	$=\{(x,0):\ x>0\}$
II :	positive vertical axis	$=\{(0,y):\ y>0\}$
III :	negative horizontal axis	$=\{(x,0):\ x<0\}$
IV :	negative vertical axis	$=\{(0,y):\ y<0\}$

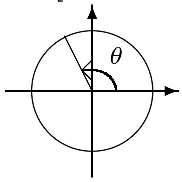


angle θ used in trigonometry is angle between radius of unit circle and positive horizontal axis measured counter clock wise naturally $\theta\in[0°,360°)$

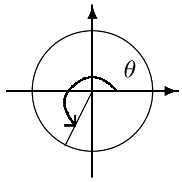




$\theta\in[0°,90°]$

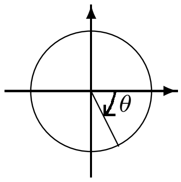


$\theta\in[90°,180°]$

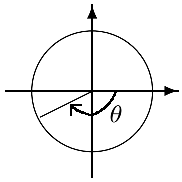


$\theta\in[180°,270°]$

Negative Angles θ are measured by $-\theta$ clockwise direction from positive horizontal axis thus same radius can give 2 different angles

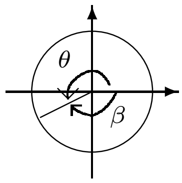


$\theta\in[-90°,0°]$



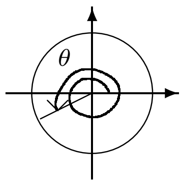
$\theta\in[-180°,-90°]$

thus same radius can give 2 different angles depending on positive or negative:



$\theta=225°$
$\beta=-135°$

$(\theta-\beta=360°)$



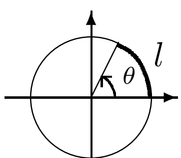
$\theta=225°+360°=585°$

Angles$>$ 360°
$\theta>360°$ is obtained by going once (or several times) around the circle (counterclockwise; for negative θ clockwise)

$\|$ The same radius corresponds to angles differing by $n\cdot 360°$. $\|$

Length of Circular arc

circumference$=2\pi$

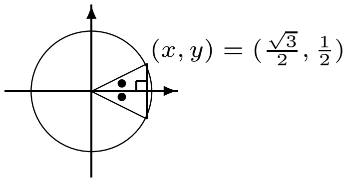


$$\frac{l}{2\pi}=\frac{\theta°}{360°}\ \longrightarrow\ l=\frac{\theta\pi}{180}$$

an angle of l radians is one with unit circle arc of length l (later more about radians)

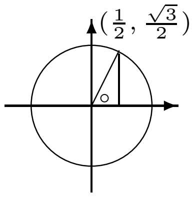
Special Points on Unit Circle

• $\theta = 30°$



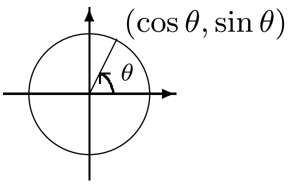
$y = \frac{1}{2} \to x = \frac{\sqrt{3}}{2}$

∘ $\theta = 60°$



θ radian	θ degree	(x, y)
$0°$	0	$(1, 0)$
$\frac{\pi}{6}$	$30°$	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
$\frac{\pi}{4}$	$45°$	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$\frac{\pi}{3}$	$60°$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$\frac{\pi}{2}$	$90°$	$(0, 1)$
π	$180°$	$(-1, 0)$

Cosine and Sine



Definition: The cosine of an angle θ, $\cos\theta$, is defined to be the first coordinate of the end point of a radius of unit circle at angle θ with positive horizontal axis.
The sine of θ, $\sin\theta$, is the end point's second coordinate.

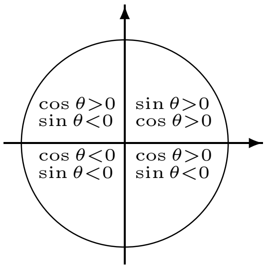
Thus the coordinates of the end point are $(\cos\theta, \sin\theta)$.

Ex.

θ degree	$\cos\theta$	$\sin\theta$
$0°$	1	0
$30°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$60°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$90°$	0	1
$180°$	-1	0

The signs of Cosine and Sine

coordinate axes divide $\mathbb{R}^2$ into quadrants(사분면).
quadrant determines sign of $\sin\theta$, $\cos\theta$

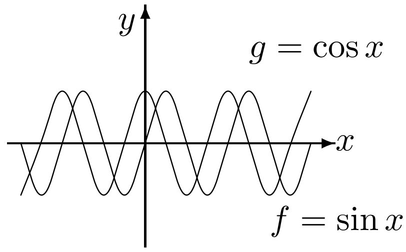


$$\begin{array}{ll} \theta \in (0, \frac{\pi}{2}) & \sin\theta > 0,\ \ \cos\theta > 0 \\ \theta \in (\frac{\pi}{2}, \pi) & \cos\theta < 0,\ \ \sin\theta > 0 \\ \theta \in (\pi, \frac{3\pi}{2}) & \sin\theta < 0,\ \ \cos\theta < 0 \\ \theta \in (\frac{3\pi}{2}, 2\pi) & \cos\theta > 0,\ \ \sin\theta < 0 \end{array}$$

<u>The Key Equation Connecting Cosine and Sine</u>
$(x, y) = (\cos\theta, \sin\theta)$ end point of radius $\in$ circle $x^2 + y^2 = 1$
$\to \ \ \underline{\cos^2\theta + \sin^2\theta = 1}$

<u>Periodicity(주기성) of sin and cos</u>
the same radius determines angles up to multiples of 2π
Thus:
$\cos(x + 2\pi) = \cos(x)$
$\sin(x + 2\pi) = \sin(x)$
<u>Definition</u>: $f :\ \mathbb{R} \to \mathbb{R}$ periodic if $\exists d > 0$ (period) with
$f(x + d) = f(x) \ \ \forall x \in \mathbb{R}$
($\forall x \in \mathrm{dom}(f) \ \ x + d \in \mathrm{dom}(f)$ also ok)
cos, sin are <u>periodic</u> functions with (minimal) period 2π.

<u>The graph of Cosine and Sine</u>
$\cos^2\theta + \sin^2\theta = 1 \quad \to \quad \cos^2\theta, \sin^2\theta \le 1$
$\to |\cos\theta|, |\sin\theta| \le 1$
Thus $\begin{array}{l} -1 \le \cos\theta \le 1 \\ -1 \le \sin\theta \le 1 \end{array}$



Domain and range of cosine and sine:
dom($cosine$)=dom(cos)=$\mathbb{R}$
range(sin)=range(cos)=$[-1, 1]$

<u>Polar coordinates</u>

$$(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\} \ \cong \ z = x + iy \in \mathbb{C} \setminus \{0\} \cong (r, \theta) \in (0, \infty) \times [0, 2\pi)$$

(x, y) Cartesian (rectangular) coordinates
(r, θ) <u>polar</u> coordinates

$r = |z|$ <u>norm</u> of $z \in \mathbb{C}$
$\theta = \arg(z)$ <u>argument</u> of z, $\qquad \theta \in [0, 2\pi)$ (or $\mathbb{R}/2\pi\mathbb{Z}$)

polar coordinates behave more naturally w.r.t. complex mult.
$\arg(z \cdot w) = \arg z + \arg w \qquad |z \cdot w| = |z| \cdot |w|$

<u>complex exponential and log</u>

$$\boxed{e^{i\theta} \;=\; \cos(\theta) + i\sin(\theta) \qquad (\theta \in \mathbb{R})}$$

thus

$$\boxed{e^{x+iy} \;=\; e^{x}\left(\cos(y) + i\sin(y)\right)}$$

Since $|\cos(y) + i\sin(y)| = 1$ and $\arg(\cos(y) + i\sin(y)) = y$ (up to $2\pi\mathbb{Z}$), we have for $z = x + iy$,

$$\begin{array}{lll} |e^{x+iy}| = e^{x} & \text{and} & \arg(e^{x+iy}) = y\,, \quad \text{or} \\ |e^{z}| = e^{\Re e z} & \text{and} & \arg(e^{z}) = \Im m z\,. \end{array}$$

One can again define a complex log as

$$\log(w) = z \text{ when } e^{z} = w \qquad (w \neq 0)\,.$$

But $\log(w)$ is defined only in $\mathbb{C}/2\pi i\mathbb{Z}$.

complex roots

Let $n \in \mathbb{N}$, $n > 1$. The multiplication formula gives

$$z^{n} \;=\; |z|^{n}\big(\cos(n\arg(z)) + i\sin(n\arg(z))\big)\,.$$

Thus when $w \neq 0$, there are n distinct n-th roots of w, i.e., complex numbers z with $z^{n} = w$.

The are given by the formula

$$z_{k+1} \;=\; \sqrt[n]{|w|}\cdot\left(\cos\left(\frac{\arg(w) + 2\pi k}{n}\right) + i\sin\left(\frac{\arg(w) + 2\pi k}{n}\right)\right)\,, \qquad k = 0,\ldots,n-1$$

In case of real roots $\sqrt[n]{w}$, $w \in \mathbb{R}$, there is a natural choice (take the positive number for $\sqrt[n]{w}$ if n is even – and $w > 0$), but for mathematicians there is no canonical choice of complex roots.

Thus the expression $\sqrt[n]{w}$ for $w \in \mathbb{C}$ is ambiguous,
and when you work with complex roots, you must say which one you mean!

0.8.3 Algebraic closedness

One main reason for importance of complex numbers is that $\mathbb{C}$ is algrabically closed.

Def F is algrabically closed field if every polynomial

$$P = z_0 + z_1 t + z_2 t^2 + \ldots + z_n t^n, \qquad z_i \in F \quad z_n \neq 0,$$

splits

$$P = z_n \cdot (t - t_0) \cdot (t - t_1) \cdot \ldots \cdot (t - t_n)\,, \qquad t_i \in F$$

t_i – root or zero of P.

<u>multiplicity</u> of $t_i = |\{ j : t_j = t_i \}| \geq 1$

(if mult$=0$, not a root)
if mult$=1$, t_i is <u>simple</u> root
if mult$=2$, t_i is <u>double</u> root
if mult$=3$, t_i is <u>triple</u> root

0.8.4 <u>Quaternions $\mathbb{H}$</u>

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R} \}$$

i, j, k formal symbols ($c = d = 0 \implies$ complex nrs) $\implies \mathbb{H} \supset \mathbb{C}$

$\curvearrowleft\cdot$	i	j	k
i	-1	$-k$	j
j	k	-1	$-i$
k	$-j$	i	-1

$$z = a + bi + cj + dk$$
$$\overline{z} = a - bi - cj - dk \qquad z \cdot \overline{z} = |z|^2 = a^2 + b^2 + c^2 + d^2$$
$$\Rightarrow z^{-1} = \frac{1}{|z|^2} \cdot \overline{z}$$

$\mathbb{H}$ forms a skew-field (like field but multiplication is non-commutative)