1. Vector spaces

1.1. Def and basic examples.

Definition 1.1. Fix a field $\mathbf{F}$. A set V with operations

$$+ : V \times V \to V \quad \text{(addition)}, \quad \text{and} \quad \cdot : \mathbf{F} \times V \to V \quad \text{(scalar mult.)},$$

is called a vector space (VS) (over '/' $\mathbf{F}$) if

$$\begin{array}{l} 1)\ (V,+) \text{ is an Abelian group} \\ 2)\ 1 \cdot \mathbf{v} = \mathbf{v}\ \forall \mathbf{v} \in V \\ 3)\ (\lambda_1\lambda_2) \cdot \mathbf{v} = \lambda_1 \cdot (\lambda_2 \cdot \mathbf{v})\ \forall \lambda_1, \lambda_2 \in F,\ \mathbf{v} \in V \ \big\} \text{ associativity} \\ \left.\begin{array}{l} 4)\ \lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}\ \forall \lambda \in F,\ \mathbf{a},\mathbf{b} \in V \\ 5)\ (\lambda_1+\lambda_2)\mathbf{a} = \lambda_1\mathbf{a} + \lambda_2\mathbf{a} \end{array}\right\} \text{distributivity} \end{array}$$

$\mathbf{x}+\mathbf{y}$ is sum of vectors $\mathbf{x}, \mathbf{y} \in V$,
$a \cdot \mathbf{x}$ is product of $a \in \mathbf{F}$, $\mathbf{x} \in V$
$\mathbf{x} \in V$ is called vector, $a \in \mathbf{F}$ a scalar

neutral element in $(V,+)$ is called 0-vector $\mathbf{0}$

Exs of vector spaces

1) $\mathbf{F}^n = \{ (a_1, \dots, a_n) : a_1, \dots, a_n \in \mathbf{F} \}$
   $(a_1, \dots, a_n)$ n-tuple, a_i elements or components of the tuple
   $\mathbf{F}^n$ is VS over $\mathbf{F}$ with the operations
   $(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$
   $a \cdot (a_1, \dots, a_n) = (a \cdot a_1, \dots, a \cdot a_n) \quad (a \in \mathbf{F})$

   vectors can be written as column vectors $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ or row vectors $(a_1, \dots, a_n)$

   in particular $(n=1)$ $\mathbf{F}$ is a VS over itself
2) the generalization of both: a matrix $m \times n$ m-rows
   n-columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \begin{array}{l} A_{ij} = a_{ij} \text{ entries} \\ a_{ii} \text{ diagonal entries} \end{array}$$

$$M_{m\times n}(\mathbf{F}) \;=\; \{\, A \,:\, A \text{ is an } m\times n \text{ matrix over } \mathbf{F}\,\} \text{ is a VS over } \mathbf{F} \text{ with}$$

$$\begin{array}{rcll} (A+B)_{ij} & := & A_{ij}+B_{ij} & \quad\text{matrix addition} \\ (cA)_{ij} & := & cA_{ij} & \quad\text{scalar mult.} \end{array}$$

(neutral element in $(M_{m\times n}(\mathbf{F}),+)$ is the <u>zero matrix</u> $\mathbf{0}_{ij}=0$.)

3) $S \neq \varnothing$ non-empty set

$$\mathcal{F}(S,\mathbf{F}) = \{\, \text{functions}\, f \,:\, S\to\mathbf{F}\,\}$$

is a VS over $\mathbf{F}$ with

$$(f+g)(s) \;=\; f(s)+g(s)\,, \quad \text{and} \quad (cf)(s) \;=\; c\cdot(f(s))$$

$\forall s\in S$.

"function" can be replaced by "continuous f." or "differentiable f." (if $\mathbf{F} = \mathbb{R}$).

4) <u>polynomial</u> over $\mathbf{F}$.

$$\mathcal{P}(\mathbf{F}) \;=\; \mathbf{F}[x] \;=\; \left\{ f(x) \;=\; \sum_{i=0}^{n} a_i x^i \quad \text{for some} \quad n\in\mathbb{N},\, a_i\in\mathbf{F},\, a_n\neq 0\right\}$$

$\deg f = n$ <u>degree</u>

$$[f]_i\,(\text{coefficient of } x^i \text{ in } f) \;=\; \begin{cases} a_i & \text{if } i\le \deg f \\ 0 & \text{otherwise} \end{cases}$$

Define add and scalar mult by

$$[f+g]_i \;=\; [f]_i+[g]_i\,, \qquad [cf]_i = c[f]_i\,.$$

Then $\mathcal{P}(\mathbf{F})$ is a VS over $\mathbf{F}$.

<u>Rem</u> There is a little difference between polynomial (with abstract variable) and a polynomial function, say $x\in\mathbb{C}\overset{f}{\longmapsto} f(x)\in\mathbb{C}$.

5) sequence in $\mathbf{F}$ $\qquad f\,:\,\mathbb{N}\to\mathbf{F}$

$(a_i)_{i=1}^{\infty} \;=\; f \;=\; (a_1,\ldots,a_n,\ldots) \qquad a_i = f(i)$ <u>sequence</u>

for $\mathbf{a}=(a_i)$ and $\mathbf{b}=(b_i)$

$\mathbf{a}+\mathbf{b} \;=\; \big((\mathbf{a}+\mathbf{b})_i\big)_{i=1}^{\infty}$ is defined by $(\mathbf{a}+\mathbf{b})_i = a_i+b_i$
$c\cdot\mathbf{a} \;=\; \big((c\cdot\mathbf{a})_i\big)_{i=1}^{\infty}$ is defined by $(c\cdot\mathbf{a})_i = c\cdot a_i$.

Theorem 1.2. *(cancellation law for vector addition)*
If for some $\mathbf{x},\mathbf{y},\mathbf{z}\in V$, $\mathbf{x}+\mathbf{z}=\mathbf{y}+\mathbf{z}$, *then* $\mathbf{x}=\mathbf{y}$.

Proof. similar to proof for $\mathbf{F}$.

Corollary 1.3. *The* $\mathbf{0}$ *vector is unique. The additive inverse* $-\mathbf{x}$ *of a vector* $\mathbf{x}$ *is unique.*

Theorem 1.4. $0 \cdot \mathbf{x} = \mathbf{0} \quad \forall \mathbf{x} \in V$

$$(-a)\cdot\mathbf{x} = -(a\cdot\mathbf{x}) \quad \forall \mathbf{x} \in V, a \in \mathbf{F}$$

$$a \cdot \mathbf{0} = \mathbf{0} \quad \forall a \in \mathbf{F}$$

1.2. Subspaces.

Let V be a VS over $\mathbf{F}$. A subset $W \subset V$ is a subspace of V if $\left(W, +\big|_{W\times W}, \cdot\big|_{\mathbf{F}\times W}\right)$ is a VS over $\mathbf{F}$.

this means

$$\boxed{\begin{array}{ll} \forall \mathbf{x},\mathbf{y} \in W & \mathbf{x}+\mathbf{y} \in W \\ \forall a \in \mathbf{F} & a\mathbf{x} \in W \end{array}} \begin{array}{l} W \text{ closed under addition} \\ W \text{ closed under scalar mult.} \end{array}$$

Theorem 1.5. *A subset $W \subset V$ is a subspace $\iff$ it is closed under addition and sc. mult. and $W \neq \varnothing$.*

Proof. $\Rightarrow$ clear $\mathbf{0} \in W \Rightarrow W \neq \varnothing$

$\Leftarrow$ we have to prove

1) $\mathbf{0} \in W$
2) $\forall \mathbf{x} \in W \quad -\mathbf{x} \in W$

other properties of VS follows from those of V.

Take $\mathbf{x} \in W$. By closedness under sc. mult. (with $0 \in \mathbf{F}$) $\mathbf{0} = 0 \cdot \mathbf{x} \in W \Rightarrow 1)$

For 2) next let $\mathbf{x} \in W$. $\exists -1 \in \mathbf{F}$ the additive inverse of mult neutral element.

By closedness under sc. mult. $W \ni -1 \cdot x \underset{\substack{\uparrow \\ \text{assoc. of } \cdot \text{ in } W}}{=} -(1\cdot\mathbf{x}) \underset{\substack{\uparrow \\ W \text{ is VS}}}{=} -\mathbf{x}$ ◻

Exs of subspaces

0) $\{\mathbf{0}\} \subset W$ is always subspace ($\overset{\text{zero}}{\text{trivial}}$ subspace); $\quad W \subset W$
1) symmetric matrices

Let $A = (A_{ij})_{i=1,j=1}^{m\ \ n}$ be $m \times n$ matrix.

We define an $n \times m$ matrix A^T, the transposed of A, by $(A^T)_{i,j} := A_{j,i}$

$$\text{example} \quad \begin{pmatrix} 1 & 2 \\ -3 & 0 \\ 4 & -5 \end{pmatrix}^T = \begin{pmatrix} 1 & -3 & 4 \\ 2 & 0 & -5 \end{pmatrix},$$

i.e. transposition interchanges rows and columns.

now we call an $n \times n$ matrix A symmetic if $A = A^T$,

$$\text{e.g.} \qquad \begin{pmatrix} 1 & 3 & 0 \\ 3 & 2 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

$\{\, A\ n \times n \text{ matrix over } \mathbf{F} : A = A^T \,\} \subset M_{n\times n}(\mathbf{F})$ subspace

2) $\mathcal{P}_n(\mathbf{F}) := \{\, P \in \mathcal{P}(\mathbf{F}) \,:\, \deg P \leq n \,\} \subset \mathcal{P}(\mathbf{F})$ subspace

3) $\{\, f : \mathbb{R} \to \mathbb{R} : f \text{ continuous} \,\} \subset \underbrace{\{ f : \mathbb{R} \to \mathbb{R} \}}_{\text{VS over } \mathbb{R}}$ is a subspace
$\{\, f : \mathbb{R} \to \mathbb{R} : f(1) = 0 \,\} \subset \ldots$

4) $A \in M_{n\times n}(\mathbf{F})$ is a diagonal matrix if $A_{ij} = 0$ for $i \neq j$

$$A = \begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}.$$

then $\{$ diag. $n \times n$ matrices over $\mathbf{F}$ $\} \subset M_{n\times n}(\mathbf{F})$
is a subspace

5) when $A = (A_{ij}) \in M_{n\times n}(\mathbf{F})$, then the trace $\mathrm{tr}(A) \in \mathbf{F}$ is defined by sum of diagonal entries

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}\,.$$

The space of traceless matrices is a subspace:

$$\{\, A \in M_{n\times n}(\mathbf{F}) \,:\, \mathrm{tr}(A) = 0 \,\} \subset M_{n\times n}(\mathbf{F})\,.$$

6) $\{A \in M_{n\times n}(\mathbf{F}) \,:\, A_{ij} \geq 0\ \forall i,j \,\} \subset M_{n\times n}(\mathbf{F})$ (with $\mathbf{F} \subset \mathbb{R}$) is not a subspace
not closed under scalar mult.

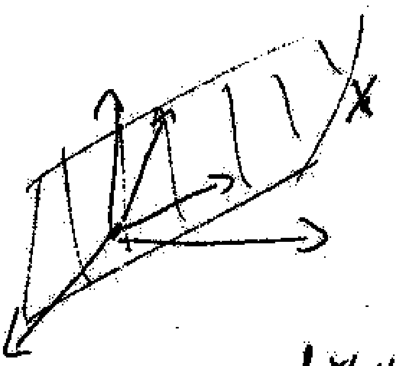
$\{\, f : \mathbb{R} \to \mathbb{R} : f(1) = 1 \,\} \subset \{ f : \mathbb{R} \to \mathbb{R} \}$
is not a subspace (not closed under addition)

The following gives a way of constructing subspaces out of others.

Theorem 1.6. *Let $W_1, W_2 \subset V$ subspaces. Then*

1) $W_1 \cap W_2$ *is a subspace of V (intersection)*
2) $W_1 + W_2 = \{\, \mathbf{w}_1 + \mathbf{w}_2 \,:\, \mathbf{w}_1 \in W_1\,,\ \mathbf{w}_2 \in W_2 \,\}$ *(sum) is a subspace of V*

1.3. Linear combinations and systems of linear equations.



X plane of v_1, v_2

consider two vectors in $\mathbb{R}^3$,

plane spanned by the vectors $\mathbf{v}_1, \mathbf{v}_2$ is of the form

$$\{ \mathbf{x} \in \mathbb{R}^3 \ : \exists \lambda_1 . \lambda_2 \in \mathbb{R} \ \ \mathbf{x} = \underset{\substack{\uparrow \\ \text{linear combination of } \mathbf{v}_1, \mathbf{v}_2}}{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2} \ \}$$

Definition 1.7. V VS over $\mathbf{F}$, $\lambda_1, \dots, \lambda_n \in \mathbf{F}$, $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$.
Then $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i$ is called <u>linear combination</u> (l.c.) of $\mathbf{v}_i$. λ_i – <u>coefficients</u> of l.c.

Example 1.8. Since $0 \cdot \mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$, the $\mathbf{0}$ vector is the linear combination of any non-empty set of vectors of V.

Sometimes it's necessary to determine whether $\mathbf{v} \in V$ is l.c. of $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$, i.e. whether $\exists \lambda_1, \dots, \lambda_n \ : \ \mathbf{v} \ = \ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$.

[e.g. given $\mathbf{x} \in \mathbb{R}^3$, does $\mathbf{x}$ lie on the plane of $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$.]

Example 1.9. $(2, 6, 8)$ l.c. of $\mathbf{u}_1 = (1, 2, 1)$, $\mathbf{u}_3 = (0, 2, 3)$
$\mathbf{u}_2 = (-2, -4, -2)$, $\mathbf{u}_4 = (2, 0, -3)$
$\mathbf{u}_5 = (-3, 8, 16)$?

$$\begin{aligned}
(2, 6, 8) &= a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_5 \mathbf{u}_5 \\
&= a_1 (1, 2, 1) + \ldots + a_5 (-3, 8, 16) \\
&= (a_1 - 2a_2 + a_4 - 3a_5, 2a_1 - 4a_2 + 2a_3 + 8a_5, \\
&\qquad a_1 - 2a_2 + 3a_3 - 3a_4 + 16a_5)
\end{aligned}$$

compare components

$$\begin{array}{cccccccl|l}
a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 & (1) & \\
2a_1 & -4a_2 & +2a_3 & & +8a_5 & = & 6 & (2) & -2\,(1) \\
a_1 & -2a_2 & +3a_3 & -3a_4 & +16a_5 & = & 8 & (3) & -\ (1)
\end{array}$$

$$\begin{array}{cccccccl|l}
a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 & & \\
 & & 2a_3 & -4a_4 & +14a_5 & = & 2 & (4) & :2 \\
 & & 3a_3 & -5a_4 & +19a_5 & = & 6 & (5) & -\frac{3}{2}\,(4)
\end{array}$$

$$\begin{array}{cccccccl} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 & \\ & & a_3 & -2a_4 & +7a_5 & = & 1 & \Big\downarrow \cdot -3 \\ & & 3a_3 & -5a_4 & +19a_5 & = & 6 & \end{array}$$

$$\begin{array}{cccccccl} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 & \\ & & a_3 & -2a_4 & +7a_5 & = & 1 & \Big\uparrow \cdot +2 \quad \Big\uparrow \cdot -2 \\ & & & a_4 & -2a_5 & = & 3 & \end{array}$$

$$\begin{array}{ccccccc} a_1 & -2a_2 & & & +a_5 & = & -4 \\ & & a_3 & & +3a_5 & = & 7 \\ & & & a_4 & -2a_5 & = & 3 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \text{solution} \\ a_1 = -4 - a_5 + 2a_2 \\ a_2 = \text{free} \\ a_3 = 7 - 3a_5 \\ a_4 = 2a_5 + 3 \\ a_5 = \text{free} \end{array}$$

By the following operations

- interchange of two rows
- mult. of an equation by <u>non-zero</u> constant
- add a multiple of an equation to <u>another</u> equation

we achieve that

- first non-zero coefficient of each equation is 1
- if unknown is first unknown with non-zero coefficient in some equation, then it does not occur in other equations
- the first unknown (with $\neq 0$ coefficient) in an equation has larger subscript than first unknown in previous equation.

Definition 1.10. Let S be a non-empty set $\subset V$, VS over $\mathbf{F}$

$$\operatorname{span}(S) = \left\{ \underbrace{\sum_{i=1}^{n} \lambda_i \mathbf{x}_i}_{\text{linear combination of } \mathbf{x}_i} : \lambda_i \in \mathbf{F},\ \mathbf{x}_i \in S \right\}$$

<u>linear span or linear hull</u> of S

<u>Properties</u>

1) $\operatorname{span}(\varnothing) = \{\mathbf{0}\}$
2) $S \subset \operatorname{span}(S)$
3) $A \subset B \Rightarrow \operatorname{span}(A) \subset \operatorname{span}(B)$
4) $\operatorname{span}(\operatorname{span}(A)) = \operatorname{span}(A)$

(2)–4): span is a <u>hull operation</u>)

5) $A = \operatorname{span}(A) \iff A$ is a linear subspace of V

$$\Big\Downarrow \text{ 3) \& 5)}$$

$(V \supset W \text{ subsp.} \supset S \Rightarrow W \supset \operatorname{span}(S))$

Definition 1.11. If $\operatorname{span}(S) = V$, we say the set S of vectors <u>spans</u> or <u>linearly generates</u> V.

1.4. Linear independence.

in the previous calculation example we saw that in the presentation $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$, the a_i are not unique.

Definition 1.12. Let $S \subset V$ VS, $S = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is linearly independent when $\forall \mathbf{v} \in V$ $\exists$ at most one $(\lambda_i)_{i=1}^{n}$: $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i$.
otherwise call S linearly dependent

Example 1.13. If $\mathbf{0} \in S$, then S is always linearly dependent, because $\lambda\mathbf{0} = \mathbf{0}$ $\forall \lambda \in \mathbf{F}$,
so uniqueness of λ_i fails for $\mathbf{v} = \mathbf{0}$.

Example 1.14. $S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3$

assume $\exists \mathbf{v} \in \mathbb{R}^3$
$$\left.\begin{aligned} \mathbf{v} &= \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 = \begin{pmatrix} 0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \\ \mathbf{v} &= \lambda_1' \mathbf{x}_1 + \lambda_2' \mathbf{x}_2 = \begin{pmatrix} 0 \\ \lambda_1' \\ \lambda_2' \end{pmatrix} \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_1 &= \lambda_1' \\ \lambda_2 &= \lambda_2' \end{aligned}$$
$\Longrightarrow$ linearly independent

Example 1.15. $S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3$ linearly dependent

Theorem 1.16. *Let $S \subset V$ be a linearly independent subset of V and $\mathbf{x} \in V$. Then $S \cup \{\mathbf{x}\}$ is linearly independent $\iff$ $\mathbf{x} \notin \operatorname{span}(S)$.*

1.5. Bases and dimension.

V VS over $\mathbf{F}$, $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$, $\lambda_1, \dots, \lambda_n \in \mathbf{F}$
called $\sum \lambda_i \mathbf{v}_i$ linear combination (of $\mathbf{v}_i$ with coefficients λ_i)

$S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent when
$\forall \mathbf{v} \in V$ $\exists$ at most 1 $(\lambda_1, \dots, \lambda_n)$: $\mathbf{v} = \sum \lambda_i \mathbf{v}_i$

S <u>generates</u>: $\iff$ $(\operatorname{span}(S) = V)$
$\forall \mathbf{v} \in V \quad \exists$ (<u>at least 1</u>) $(\lambda_1, \dots, \lambda_n)$: $\quad \mathbf{v} = \sum \lambda_i \mathbf{v}_i$

Definition 1.17. If S is linearly independent and generating, then call S a <u>basis</u> of V.
$\big(\iff \forall \mathbf{v} \in V \quad \exists! \ (\lambda_1, \dots, \lambda_n) : \quad \mathbf{v} = \sum \lambda_i \mathbf{v}_i\big)$

Example 1.18. $\operatorname{span}(\varnothing) = \{\mathbf{0}\}$ and $\varnothing$ is linearly independent $\Longrightarrow \varnothing$ is a basis of $\{\mathbf{0}\}$

Example 1.19. $\mathbf{F}^n \supset \{\mathbf{e}_i\}_{i=1}^n \quad \mathbf{e}_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$
standard basis

Example 1.20. $M_{m\times n}(\mathbf{F})$. Let E^{ij} for $\begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array}$ be the matrix

$$(E^{ij})_{kl} = \underbrace{\delta_{ik}\delta_{jl}}_{\text{Kronecker's delta}} \qquad E^{ij} = \begin{array}{c c} & j \\ i & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array}.$$

then $\{E^{ij}\}_{i=1, j=1}^{m, \ \ n}$ is a basis for $M_{m\times n}(\mathbf{F})$.

Example 1.21. $\mathcal{P}_n(\mathbf{F}) = \{$ polynomials in $\mathcal{P}(\mathbf{F})$ of degree $\le n\}$
$S = \{1, x, x^2, \dots, x^n\}$ standard basis.

Example 1.22. $\mathcal{P}(\mathbf{F}) \qquad S = \{1, x, x^2, \dots, x^n, \dots\}$.

<u>Rem.</u> when S is infinite, we define a linear combination of elements in S by $\sum_{i=1}^{n} \lambda_i \mathbf{x}_i$ where n is arbitrary large but $< \infty$ and $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ any subset of n elements of S.

Theorem 1.23. *Let $S = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be a <u>finite</u> generating set of VS of V. Then $\exists$ basis $S' \subseteq S$ of V.*

Proof. If $V = \{\mathbf{0}\}$, then $S' = \varnothing \subseteq S$ basis.
So assume $V \neq \{\mathbf{0}\}$. Then $\exists \mathbf{x}_1 \ni S \quad \mathbf{x}_1 \neq \mathbf{0}$ (order $\mathbf{x}_i$ properly);
then $S_1 = \{\mathbf{x}_1\}$ is linearly independent.

We construct now sets S_i with $\quad S_i \subset \{\mathbf{x}_1, \dots, \mathbf{x}_i\}$
then $S' := S_n$ is the basis we sought. $\quad \operatorname{span}(S_i) = \operatorname{span}(\{\mathbf{x}_1, \dots, \mathbf{x}_i\})$
S_i linearly independent.

$i = 1$ done; For $i = 1, \ldots, n-1$ do the following:

if $\mathbf{x}_{i+1} \in \operatorname{span}(S_i)$, set $S_{i+1} := S_i$,
else set $S_{i+1} = S_i \cup \{\mathbf{x}_{i+1}\}$

Claim 1 $\operatorname{span}(S_{i+1}) = \operatorname{span}(\{\mathbf{x}_1, \ldots, \mathbf{x}_{i+1}\})$.

pf. if $\mathbf{x}_{i+1} \in \operatorname{span}(S_i)$,

$$\begin{aligned} \text{then } \operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_{i+1}) &= \operatorname{span}(\operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_i) \cup \{\mathbf{x}_{i+1}\}) \\ &= \operatorname{span}(S_i, \mathbf{x}_{i+1}) \\ &\underset{\substack{\uparrow \\ \mathbf{x}_{i+1} \in \operatorname{span}(S_i)}}{=} \operatorname{span}(S_i) = \operatorname{span}(S_{i+1}). \end{aligned}$$

if $\mathbf{x}_{i+1} \notin \operatorname{span}(S_i)$,
$\operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_{i+1}) = \operatorname{span}(S_i \cup \{\mathbf{x}_{i+1}\}) = \operatorname{span}(S_{i+1})$. □

Claim 2 S_{i+1} is linearly independent.

pf if $\mathbf{x}_{i+1} \in \operatorname{span}(S_i)$,
$S_{i+1} = S_i$ linearly independent.

if $\mathbf{x}_{i+1} \notin \operatorname{span}(S_i)$,
then $S_i \cup \{\mathbf{x}_{i+1}\}$ linear independent by theorem 1.16. □

Theorem 1.24. *(Replacement theorem)*
Let V be a VS generated by G with $|G| = n$. Let L be a linearly independent subset of V with $|L| = m$.
Then $m \leq n$, and $\exists H \subseteq G$ with $|H| = n - m$ such that $\operatorname{span}(L \cup H) = V$.

Corollary 1.25. *Let V have a basis G and $|G| = n < \infty$, and let G' be a different basis $\Longrightarrow |G'| = n$.*

Proof. Take $L = G'$ in previous theorem
it asserts that $|L| = m \leq n$.
Reverse role of G and G' $\Rightarrow m \geq n \quad \Longrightarrow m = n$ □

Definition 1.26. Dimension of a VS V, $\dim(V)$ is cardinality of a basis if $\exists$ finite generating set; else $\dim(V) = \infty$.

when $\dim(V) = \infty$ V is infinite dimensional
$\dim(V) < \infty$ V is finite dimensional

Ex. $\dim\{\mathbf{0}\} = 0.$

Ex. $\dim \mathbf{F}^n = n$

Ex. $\dim M_{m\times n}(\mathbf{F}) = mn$

Ex. $\dim \mathcal{P}_n(\mathbf{F}) = n+1$

Ex. $\dim_{\mathbb{C}} \mathbb{C} = 1$ (basis $\{1\}$)
$\dim_{\mathbb{R}} \mathbb{C} = 2$ (basis $\{1, i\}$)
$\dim_{\mathbb{R}} \mathbb{R} = 1$ but $\dim_{\mathbb{Q}} \mathbb{R} = \infty$ (not easy to prove)

$\dim_{\mathbb{Q}} \mathbb{R} = \infty$ is related to the existence of transcendental numbers
$\alpha \in \mathbb{R}$ transcendental$: \iff P(a) \neq 0$ for all $P \in \mathcal{P}(\mathbb{Q})$.

α transcendental $\iff \{1, \alpha, \alpha^2, \dots\} \subset \mathbb{R}$ is linearly independent over $\mathbb{Q}$

Rem α is irrational $\iff P(a) \neq 0$ for all $P \in \mathcal{P}_1(\mathbb{Q})$
$\iff \{1, \alpha\} \subset \mathbb{R}$ is linearly independent over $\mathbb{Q}$

It is known, e.g., that π and e are transcendental; $\sqrt{2}$ is irrational but not transcendental.

Proposition 1.27. $\dim \mathcal{P}(\mathbf{F}) = \infty$.

Proof. Assume $\dim \mathcal{P}(\mathbf{F}) = n < \infty$.
Then as a consequence of Replacement theorem, any linearly independent set $S \subset \mathcal{P}(\mathbf{F})$ has $m \leq n < \infty$ elements.
But $\mathcal{P}(\mathbf{F})$ has the linearly independent set $S = \{1, x, x^2, \dots\}$ with $|S| = \infty$ ↯.
So $\dim \mathcal{P}(\mathbf{F}) = \infty$. $\square$

Remark 1.28. We prove that when $\dim(V) < \infty$, then V has a basis. This is true also when $\dim(V) = \infty$, but its proof depends on deep logic (Axiom of choice) and I will not do it in class (see §1.7 in book).

Corollary 1.29. *V VS over $\mathbf{F}$,* $n = \dim(V) < \infty$.

(a) *any finite generating set S of V has $|S| \geq n$,*
S is a basis $\iff |S| = n$
(b) *a linearly independent set S of V has $|S| \leq n$.*
S is a basis $\iff |S| = n$
(c) *every linearly independent set S can be extended to a basis*

Recall: (d) *every spanning subset S can be reduced to a basis*

Example 1.30. show that $x^2+3x-2,\ 2x^2+5x-3,\ x^2-4x+4$ is a basis of $\mathcal{P}_2(\mathbb{R})$.

$\dim \mathcal{P}_2(\mathbb{R}) = |S| = 3$, so enough to prove S is generating

$$\begin{aligned} ax^2+bx+c &= (-8a+5b+3c)(x^2+3x-2) \\ &+ (4a-2b-c)(2x^2+5x-3) \\ &+ \underset{\substack{\uparrow \\ \text{how to find this we discussend when we talked about linear eqn systems}}}{(-a+b+c)}(-x^2-4x+4) \end{aligned}$$

Example 1.31. show that $S = \left\{ \underset{}{\overset{M_1}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}}, \overset{M_2}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \overset{M_3}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}, \overset{M_4}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \right\}$

is a basis of $M_{2\times 2}(\mathbf{F})$ (assume $\mathrm{char}(\mathbf{F}) \neq 3$).

Proof. we prove S generates $\qquad 4 = |S| = \dim M_{2\times 2}$

options: 1) solve linear eqn system

2) (better) enough to prove $\underbrace{\overset{E_1}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}, \overset{E_2}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \overset{E_3}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{E_4}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}}_{S'} \in \mathrm{span}(S)$

because $S' \subset \mathrm{span}(S) \Longrightarrow$

$V = \mathrm{span}(S') \subset \mathrm{span}(\mathrm{span}(S)) = \mathrm{span}(S) \subset V$

$\Longrightarrow \mathrm{span}(S) = V$.

$$M_1 + M_2 + M_3 + M_4 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

$$\underbrace{M_3 - \frac{1}{3}(M_1+M_2+M_3+M_4)}_{\downarrow} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$E_3 = -\Big(\qquad\qquad\qquad \Big)$$

$$E_i = \underbrace{-(M_i - \frac{1}{3}(M_1+M_2+M_3+M_4))}_{\text{l.c. of } M_i}$$

□

<u>Dimension of subspaces</u>

Theorem 1.32. *Let V be a VS over $\mathbf{F}$, $W \subseteq V$ subspace*
Then $\dim(W) \leq \dim(V)$ & (if $\dim V < \infty$!) "=" $\iff$ $V = W$.

Proof. A basis S' of W is linearly independent in V, thus by corollary 1.29, part c) $\exists S \supseteq S'$ basis of V.

$\dim(V) = |S| \geq |S'| = \dim(W)$.

Assume $|S| = |S'|$. Then $S \supseteq S' \Rightarrow S = S'$
so $S' = S$ is a basis of V.

$$W \underset{\substack{\uparrow \\ S' \text{ basis of } W}}{=} \mathrm{span}(S') \underset{\substack{\uparrow \\ S = S'}}{=} \mathrm{span}(S) \underset{\substack{\uparrow \\ S \text{ basis of } V}}{=} V$$

□

<u>Ex.</u>

$$1)\ \left\{\ \underset{\substack{\| \\ DM_{n\times n}(\mathbf{F})}}{\text{diag matrices}}\quad \operatorname{diag}(a_1,\ldots,a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \right\} \subseteq M_{n\times n}(\mathbf{F}).$$

basis for $DM_{n\times n}(\mathbf{F})$ $\left\{ \underset{\substack{\| \\ \begin{array}{cc} & i \\ i & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \\ 0 & & 0 \end{pmatrix} \end{array}}}{E^{ii}} \ :\ 1 \le i \le n \right\}$.

then $\dim DM_{n\times n}(\mathbf{F}) = n < n^2 = \dim M_{n\times n}(\mathbf{F})$

2) $\{ M \in M_{n\times n}(\mathbf{F}) : M = M^T \} \subseteq M_{n\times n}(\mathbf{F})$

symmetric matrices

basis $\{E^{ij} + E^{ji},\quad 1 \le i < j \le n\} \cup \{E^{ii} : 1 \le i \le n\}$

$|\text{basis}| = \dfrac{n(n+1)}{2} < n^2 = \dim M_{n\times n}(\mathbf{F})$

Lagrange interpolation

$c_0, c_1, \ldots, c_n \in \mathbf{F}$ scalars. consider

$$f_i(x) = \prod_{\substack{k=0 \\ k \ne i}}^{n} \frac{(x - c_k)}{(c_i - c_k)}$$

$f_i(c_j) = \delta_{ij} \Longrightarrow f_i$ are linearly independent

reason is : given $f : \mathbf{F} \to \mathbf{F}$

$$\text{consider} \quad \sum f(c_i) f_i = \tilde{f} \in \mathcal{P}_n(\mathbf{F}) \tag{6}$$

f and $\tilde{f}$ have the same values in $\{c_i\}_{i=0}^{n}$

$$\| $$

$$\underset{\substack{\uparrow \\ \text{polynomial}}}{\tilde{f}} \text{ approximates } \underset{\substack{\uparrow \\ \text{arbitrary function}}}{f}$$

(6) is called Lagrange interpolation