2. Linear transforamtions and matrices

Motivation: want to study linear maps between VS
that "preserve" the VS structure
these occur in
- calculus (differentiation, intergation)
- geometry (rotations, reflactions, projections)

$\Longrightarrow$ linear maps/transformations

Assumption: all VS considered over a common field $\mathbf{F}$. (Fixed one for the whole §2.)

2.1. linear transformations, null spaces, and ranges.

Recall $T : V \to W$ denotes function/map from V to W.

Definition 2.1. Let V, W be VS (/F)
$T : V \to W$ is a linear transformation (from V to W)
limply linear if for all $\mathbf{x}, \mathbf{y} \in V$, $c \in \mathbf{F}$:

(a) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ (additive)
(b) $T(c\mathbf{x}) = cT(\mathbf{x})$

(If $\mathbf{F} = \mathbb{Q}$, then (a)$\Longrightarrow$(b), i.e., each additive map is linear, but not in general.)

Properties of a linear map T

(1) $T(\mathbf{0}) = \mathbf{0}$
(2) T linear $\iff \forall c \in \mathbf{F},\ \mathbf{x}, \mathbf{y} \in V$
$T(c\mathbf{x}+\mathbf{y}) = cT(\mathbf{x}) + T(\mathbf{y})$

(3) If T is linear, then $T(\mathbf{x}-\mathbf{y}) = T(\mathbf{x}) - T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$
(4) T linear $\iff \forall \mathbf{x}_1, \ldots, \mathbf{x}_n \in V, \quad c_1, \ldots, c_n \in \mathbf{F}$,

$$T\left(\sum_{i=1}^{n} c_i \mathbf{x}_i\right) = \sum_{i=1}^{n} c_i\, T(\mathbf{x}_i)$$

Example 2.2. $T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T(a_1, a_2) = (2a_1 + a_2, a_1)$

$$T(c(a_1, a_2)) = T(ca_1, ca_2) = (2ca_1 + ca_2, ca_1) = c(2a_1 + a_2, a_1) = cT(a_1, a_2)$$

$$T((a_1, a_2) + (b_1, b_2)) = T(a_1 + b_1, a_2 + b_2) = (2(a_1 + b_1) + (a_2 + b_2), a_1 + b_1) = (2a_1+a_2, a_1)+(2b_1+b_2, b_1) = T(a_1, a_2)+T(b_1, b_2)$$

3 important examples from geometry

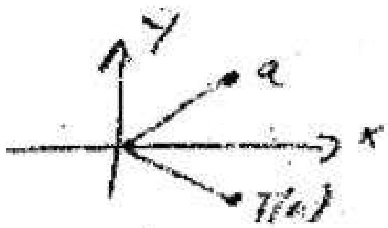
Example 2.3.  rotation by θ

$$T(a_1,a_2) = (b_1,b_2) = (a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta)$$

$$\begin{aligned} b_1 + b_2 i &= e^{i\theta}(a_1 + a_2 i) \\ &= (\cos\theta + i\sin\theta)(a_1 + a_2 i) \\ &= (a_1\cos\theta - a_2\sin\theta) \\ &\quad +(a\sin\theta + a_2\cos\theta)i \end{aligned}$$

Exercise: show that T is linear.

Example 2.4. $T(a_1,a_2) = (a_1,-a_2)$
reflection (along x-axis)
it satisfies $T^2 = Id_V \leftarrow$ identity



Example 2.5. $T(a_1,a_2) = (a_1,0)$
projection (onto x-axis)
$T^2 = T$

Example 2.6. (now matrices)
$T : M_{n\times n}(\mathbf{F}) \to M_{n\times n}(\mathbf{F}) \qquad T(A) = A^T$
$(A+B)^T = A^T + B^T \quad (cA)^T = cA^T \Longrightarrow T$ is linear.

Example 2.7. $V = \{ f : \mathbb{R} \to \mathbb{R} : f \in C^\infty \}$ or $V = \mathcal{P}(\mathbb{R}) = \mathbb{R}[z]$.

$T : V \to V \quad T(f) = f'$. The differential operator is linear.

Example 2.8. $T(f) = \int_0^x f(t)\,dt$

Definition 2.9. V, W VS /F $\quad T : V \to W$ linear

$\ker(T) = N(T) = \{ \mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0} \} = T^{-1}(\mathbf{0})$ (with $\mathbf{0} \in W$) nullspace or kernel of T

$\mathrm{Im}(T) = R(T) = \{ T(\mathbf{v}) \in W : \mathbf{v} \in V \} \subset W$ range or image of T

Example 2.10. Let $V = W$ and $T = Id_V$ $T(\mathbf{v}) = \mathbf{v}$ identity map
Then $\ker T = \{\mathbf{0}\}$ $\mathrm{Im}T = V$
Let $T = 0$ $T(\mathbf{v}) = \mathbf{0} \in W$ $\forall \mathbf{v} \in V$ zero map
Then $\ker T = V$ $\mathrm{Im} = \{\mathbf{0}\}$

Theorem 2.11. *Let $T : V \to W$ linear*
Then $\ker T \subset V$ and $\mathrm{Im}T \subset W$ are linear subspaces.

Theorem 2.12. *Let $T : V \to W$ be linear*
and let $S \subset V$ be a subset.
Then $T(\mathrm{span}(S)) = \mathrm{span}(T(S))$.

Corollary 2.13. *in particular when S generating ($\iff$ $\mathrm{span}(S) = V$)*
then $\mathrm{span}(T(S)) = \mathrm{Im}T$.

Corollary 2.14. *S basis* $\Rightarrow \dim \mathrm{Im}T \leq \dim V$

Definition 2.15. $T : V \to W$ linear
$\dim(\ker(T)) =: \mathrm{nul}(T)$ nullity
$\dim(\mathrm{Im}(T)) =: \mathrm{rk}\,(T)$ rank of T

Theorem 2.16. *(Dimension theorem)*
Let $T : V \to W$ linear, $\dim(V) < \infty$.

$$\textit{Then} \qquad \mathrm{nul}(T) + \mathrm{rk}\,(T) \;=\; \dim(V)\,. \tag{1}$$

Theorem 2.17. $T : V \to W$ $\overset{\textit{injective}}{\textit{1-to-1}}$ $\iff$ $\ker(T) = \{\mathbf{0}_V\}$.

Proof. 1) If T injective $\Rightarrow |T^{-1}(\mathbf{y})| \leq 1\ \forall \mathbf{y} \in W$
Take $y = \mathbf{0}_W$ $|T^{-1}(\mathbf{0})| \leq 1$,
but $\mathbf{0}_V \in T^{-1}(\mathbf{0}_W)$ so $T^{-1}(\mathbf{0}_W) = \{\mathbf{0}_V\}$.
2) Assume $\ker(T) = \{\mathbf{0}\}$. Let $T(\mathbf{v}) = T(\mathbf{v}') \in W$
for some $\mathbf{v}, \mathbf{v}' \in V$,
then because T is linear $T(\mathbf{v} - \mathbf{v}') = T(\mathbf{v}) - T(\mathbf{v}') = \mathbf{0}$,
so $\mathbf{v} - \mathbf{v}' \in \ker T$. But $\ker T = \{\mathbf{0}\}$, so $\mathbf{v} - \mathbf{v}' = \mathbf{0}$
$\Longrightarrow \mathbf{v} = \mathbf{v}'$. □

Theorem 2.18. *$T : V \to W$ linear, $n = \dim(V) = \dim(W) < \infty$. The following are equivalent.*

(a) *T injective ($\iff$ $\mathrm{nul}(T) = 0$)* $\iff$
(b) *T surjective ($\iff$ $\mathrm{rk}\,(T) = n$)*

Proof. (a) T injective $\mathrm{nul}(T) = \dim\ker T = \dim\{\mathbf{0}\} = 0$

$$\text{theorem 2.16} \Rightarrow \ \mathrm{rk}\,(T) = \dim V = \dim W = n \tag{2}$$

(2) $\Rightarrow \dim \mathrm{Im}T = \dim W$; $\mathrm{Im}T \subseteq W$ subsp $\Rightarrow \mathrm{Im}T = W \Rightarrow T$ surjective

(b) (a) $\Rightarrow \operatorname{rk}(T) = n = \dim V \underset{\text{Th 2.16}}{\Longrightarrow} \operatorname{nul}(T) = 0 \Rightarrow \ker T = \{\mathbf{0}\}$ □

Example 2.19. $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R}) \quad f(x) = a_0 + a_1 x + a_2 x^2$

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt.$$

$$\begin{aligned} R(T) &= \operatorname{span}\{T(1), T(x), T(x^2)\} \\ &= \operatorname{span}\Big\{\underbrace{3x, 2 + \frac{3}{2}x^2, 4x + x^3}_{\text{linearly independent}}\Big\} \end{aligned}$$

$$\operatorname{rk}(T) = \dim \overset{\operatorname{Im}(T)}{R(T)} = 3 < 4 = \dim \mathcal{P}_3(\mathbb{R}) \quad (T \text{ \underline{not surjective}})$$

$$\underset{\underset{3}{\|}}{\operatorname{nul}(T) + \operatorname{rk}(T)} = \underset{\underset{3}{\|}}{\dim} \overset{\overset{\mathcal{P}_2(\mathbb{R})}{\|}}{V} \Longrightarrow \operatorname{nul}(T) = 0 \Longrightarrow T \text{ \underline{injective}}$$

Theorem 2.20. *If V, W VS /$\mathbf{F}$, $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ basis of V, then for each $(\mathbf{w}_1, \ldots, \mathbf{w}_n) \in W^n$ $\exists!$ linear $T : V \to W$ with $T(\mathbf{v}_i) = \mathbf{w}_i$.*

Corollary 2.21. *Let $T, U : V \to W$ be linear maps*
$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be basis of V.
If $T(\mathbf{v}_i) = U(\mathbf{v}_i)$ $\forall i = 1, \ldots, n \Longrightarrow T = U$.
$T\Big|_S = U\Big|_S$

(Linear maps are identified by their images on a basis.)

2.2. Matrix representations of linear transformations.

fix ordered bases (OB) $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V and $\gamma = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of W.

Example 2.22. $(\mathbf{e}_i)_{i=1}^n$ ordered basis of $\mathbf{F}^n$; standard ordered basis (SOB).

Definition 2.23. If $\beta = (\mathbf{v}_i)_{i=1}^n$ is ordered basis of V
then $\forall \mathbf{x} \in V \quad \exists!(a_1, \ldots, a_n) : \mathbf{x} = \sum_{i=1}^n a_i \mathbf{v}_i$.
$(a_1, \ldots, a_n)$ is called coordinate vector $[\mathbf{x}]_\beta$ of $\mathbf{x}$ w.r.t. the basis β.

Example 2.24. $\mathbf{x} = (1, 3, 2) \in \mathbb{R}^3 \qquad \beta = (\mathbf{e}_i)_{i=1}^3$
$\mathbf{x} = \mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3 \Rightarrow [\mathbf{x}]_\beta = (1, 3, 2) = \mathbf{x}$.

$\mathbf{x}$ is equal to its coordinate vector w.r.t. β;

thus we denote elements in $\mathbf{F}^n$ by their coordinate vector w.r.t. the SOB

Example 2.25. $V = \mathcal{P}_2(\mathbb{R})$ $\quad S = (1, x, x^2)$

$$f(x) = 4 + 6x - 7x^2$$

$[f]_S = (4, 6, -7)$

Now let $T : V \to W$ linear, and fix ordered bases $\underset{(\mathbf{v}_i)}{\beta}, \underset{(\mathbf{w}_j)}{\gamma}$ of V, W.

$T(\mathbf{v}_i) = \sum_{j=1}^{m} a_{ji}\mathbf{w}_j \iff [T(\mathbf{v}_i)]_\gamma = (a_{ji})_{j=1}^{m}$ (note that a_{ij}uniquely determine T)

Define the <u>matrix</u> $A \in M_{m\times n}(\mathbf{F})$ by $A_{ji} = a_{ji}$ $\quad i = 1, \dots, n \quad j = 1, \dots, m$

Then $A = [T]_\beta^\gamma$ is the <u>matrix presentation</u> of T w.r.t. the ordered bases β of V, γ of W.
(If $V = W$ and $\beta = \gamma$, write $[T]_\beta$ for $[T]_\beta^\beta$.)

Example 2.26. $T : \mathbb{R}^2 \to \mathbb{R}^3$ (with β, γ standard bases)

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$$

$T(\mathbf{e}_1) = T(1, 0) = 1\mathbf{e}_1 + 0\mathbf{e}_2 + 2\mathbf{e}_3 = (\boxed{1,0,2})$ $\quad a_{11} = 1$, $a_{21} = 0$, $a_{31} = 2$

$T(\mathbf{e}_2) = T(0, 1) = 3\mathbf{e}_1 + 0\mathbf{e}_2 - 4\mathbf{e}_3 = (\boxed{3, 0, -4})$ $\quad a_{12} = 3$, $a_{22} = 0$, $a_{32} = -4$

$$A = [T]_\beta^\gamma = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$

If $\gamma' = (\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1)$, then

$$\begin{aligned} [T(\mathbf{e}_1)]_{\gamma'} &= (2, 0, 1) \\ [T(\mathbf{e}_2)]_{\gamma'} &= (-4, 0, 3) \end{aligned} \implies [T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}.$$

Example 2.27. $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ $\quad T(f) = f'$ $\qquad [T]_\beta^\gamma := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$\beta = (1, x, x^2, x^3)$, $\quad \gamma = (1, x, x^2)$ $\quad T(x) = 1$ $\quad [T(x)]_\gamma = (1, 0, 0)$

$T(1) = 0 \Rightarrow [T(1)]_\gamma = (0, 0, 0)$ $\quad T(x^2) = 2x$ $\quad [T(x^2)]_\gamma = (0, 2, 0)$

$T(x^3) = 3x^2$ $\quad [T(x^3)]_\gamma = (0, 0, 3)$

Definition 2.28. V, W VS over $\mathbf{F}$.

$$\mathcal{L}(V, W) = \{ T : V \to W : \quad T \text{ linear} \}$$

the space of linear maps between V and W.

When $V = W$ set $\mathcal{L}(V,V) =: \mathcal{L}(V)$.

Theorem 2.29. *$\mathcal{L}(V,W)$ is a VS over $\mathbf{F}$ with the operations*

$$\begin{aligned}(T+U)(\mathbf{v}) &:= T(\mathbf{v}) + U(\mathbf{v}) \qquad (\mathbf{v} \in V)\\ (cT)(\mathbf{v}) &:= cT(\mathbf{v})\end{aligned}$$

Moreover, for fixed bases β, γ of V, W, we have

$$\begin{aligned}[T+U]_\beta^\gamma &= [T]_\beta^\gamma + [U]_\beta^\gamma\\ [cT]_\beta^\gamma &= c[T]_\beta^\gamma\end{aligned}$$

$\Rightarrow$

adding and scalar mult. of linear maps	corresponds to	adding and scalar mult. of matrices

So matrices are a way to calculate with linear maps – this is why they are useful!

2.3. composition of linear maps and matrix multiplication.

We learned $T : V \to W \qquad U : W \to Z$

$$\exists U \circ T : V \to Z \quad U \circ T(\mathbf{x}) = U(T(\mathbf{x}))$$

Theorem 2.30. *$T : V \to W \quad U : W \to Z$ are linear $\Longrightarrow$ $U \circ T : V \to Z$ is linear*

Proof. $U \circ T(\mathbf{a}+\mathbf{b}) = U(T(\mathbf{a}+\mathbf{b})) \underset{T \text{ is linear}}{=} U(T(\mathbf{a}) + T(\mathbf{b})) \underset{U \text{ is linear}}{=} U(T(\mathbf{a})) + U(T(\mathbf{b}))$

$$= U \circ T(\mathbf{a}) + U \circ T(\mathbf{b})$$ □

Theorem 2.31. *V VS /$\mathbf{F}$ T, U_1, $U_2 \in \mathcal{L}(V)$. Let $\boxed{TU = T \circ U}$.*

$T(U_1+U_2) = TU_1 + TU_2 \quad (U_1+U_2)T = U_1T + U_2T$ *distributive*

$T(U_1U_2) = (TU_1)U_2$ *associative*

$TI = IT \qquad I = Id_V$ *multiplicative identity*

$a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$

$\mathcal{L}(V)$ is almost a field except that

1) multiplication is non-commutative
2) $\exists U \neq 0$ with no multiplicative inverse (unless $\dim V = 1$ and $\mathcal{L}(V) \simeq \mathbf{F}$)

$\mathcal{L}(V)$ is "ring" (ex: $\mathbb{Z}$ is a ring),
but non-commutative

How to define matrix multiplication to correspond to composition of maps?

$$\begin{array}{llll} T & : V \longrightarrow W & & U : W \longrightarrow Z \\ \text{bases} & \alpha \qquad \beta & & \beta \qquad \gamma \\ & \| \qquad \| & & \qquad \| \\ & \{\mathbf{v}_i\}_{i=1}^n \ \{\mathbf{w}_j\}_{j=1}^m & & \qquad \{\mathbf{z}_k\}_{k=1}^p \end{array}$$

$A = [U]_\beta^\gamma \quad B = [T]_\alpha^\beta$ $\qquad$ $B = m \times n$-matrix

$\underset{C}{\overset{AB}{\|}} = [UT]_\alpha^\gamma$ $\qquad$ $A = p \times m$-matrix

$C = p \times n$-matrix

$$\begin{aligned} UT(\mathbf{v}_i) \ &= \ U(T(\mathbf{v}_i)) \ = \ U\Big(\sum_{j=1}^m B_{ji}\mathbf{w}_j\Big) \\ &= \sum_{j=1}^m B_{ji}U\big(\mathbf{w}_j\big) \ = \ \sum_{j=1}^m B_{ji}\Big(\sum_{k=1}^p \ A_{kj}\mathbf{z}_k\Big) \\ & \sum_{k=1}^p\Big(\sum_{j=1}^m A_{kj}B_{ji}\Big)\mathbf{z}_k \ = \ \sum_{k=1}^p C_{ki}\mathbf{z}_k \end{aligned}$$

where $\boxed{C_{ki} \ = \ \sum_{j=1}^m A_{kj}B_{ji}}$

Definition 2.32. Let $A \in M_{p\times m}(\mathbf{F}) \quad B \in M_{m\times n}(\mathbf{F})$, then define $AB \in M_{p\times n}(\mathbf{F})$ by

$$(AB)_{ij} \ = \sum_{k=1}^m A_{ik}B_{kj} \qquad 1 \le i \le p, \quad 1 \le j \le n$$

This is called product AB of A and B.

Ex. $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1\cdot 4 + 2\cdot 2 + 1\cdot 5 \\ 0\cdot 4 + 4\cdot 2 + (-1)\cdot 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \end{pmatrix}.$

Theorem 2.33. *Let V, W, Z be finite dim. VS with bases α, β, γ.*
Let $T : V \to W \quad U : W \to Z$ be linear.
Then $[UT]_\alpha^\gamma \ = \ [U]_\beta^\gamma \ [T]_\alpha^\beta$.

Corollary 2.34. *(when $V = W = Z$) V VS $/\mathbf{F}$ finite dim.*

$$T, U \in \mathcal{L}(V) \qquad [UT]_\beta \ = \ [U]_\beta[T]_\beta$$

Ex. $U : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}) \qquad T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$

$U(f) = f' \qquad T(f) = \int_0^x f(t)\ dt$

Let α and β be SOB of $\mathcal{P}_3$ and $\mathcal{P}_2$ resp.

$$[UT]_\beta \quad = \quad [U]_\alpha^\beta [T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [I]_\beta$$

Definition 2.35. $I_n \in M_{n\times n}(\mathbf{F})$ with $(I_n)_{ij} = \delta_{ij}$ (Kronecker's delta) is called <u>identity matrix</u>

$$I_1 = (1), \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem 2.36. *V, W f.d. VS /$\mathbf{F}$ β, γ ordered bases, $\mathbf{u} \in V$*
$\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta$$

In particular, the i-th column of $[T]_\beta^\gamma$ is $[T(\mathbf{v}_i)]_\gamma$.

Definition 2.37. $A \in M_{m\times n}(\mathbf{F})$ define

$L_A : \mathbf{F}^n \longrightarrow \mathbf{F}^m$ defined by $L_A(\mathbf{x}) = A\mathbf{x}$

left-multiplication transformation

example $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ 2×3 matrix $L_A : \mathbb{R}^3 \to \mathbb{R}^2$

let β be SOB of $\mathbb{R}^3$, γ SOB of $\mathbb{R}^2$.

$$\underset{\underset{[(1,0,0)]_\beta}{\|}}{L_A\,(1,0,0)} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$L_A(0,1,0) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \Rightarrow [L_A]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$

$$L_A(0,0,1) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_\gamma$$

Let $A \in M_{m\times n}(\mathbf{F})$. Then $L_A : \mathbf{F}^n \to \mathbf{F}^m$ linear. If β, γ SOB (順基底) of $\mathbf{F}^n$, $\mathbf{F}^m$.

Theorem 2.38. (a) $[L_A]_\beta^\gamma = A$

(b) $L_A = L_B \iff A = B$

(c) $L_{A+B} = L_A + L_B$ and $L_{aA} = aL_A$ for all $a \in \mathbf{F}$.

(d) If $T : \mathbf{F}^n \to \mathbf{F}^m$ is linear, $T = L_{[T]_\beta^\gamma}$

(e) If $E \in M_{n\times p}(\mathbf{F})$ $\quad \big(L_E : \mathbf{F}^p \to \mathbf{F}^n\big)$
then $L_{AE} = L_A L_E$
(f) If $m = n$, then $L_{I_n} = Id_{\mathbf{F}^n}$

Corollary 2.39. *Let A, B, C be matrices, such that $A(BC)$ is defined. Then $(AB)C$ is also defined, and $A(BC) = (AB)C$.*

Proof. By part (e) of Theorem 2.38, and associativity of $\circ$ for maps

$$L_{A(BC)} = L_A \circ L_{BC} = L_A \circ (L_B \circ L_C) = (L_A \circ L_B) \circ L_C = L_{AB} \circ L_C = L_{(AB)C}$$

now by part (b) of Theorem 2.38 $\Longrightarrow$ □

2.4. Invertibility and isomorphism.

Definition 2.40. A linear map $T : V \to W$ is invertible if
$\exists\, U : W \to V$ with $TU = Id_W$ and $UT = Id_V$.
$U = T^{-1}$ inverse

Theorem 2.41. *T invertible $\iff$ T is bijective*
in this case T^{-1} is unique
moreover, $(TU)^{-1} = U^{-1}T^{-1}$ and $(T^{-1})^{-1} = T$.

Theorem 2.42. *$T : V \to W$ invertible linear*
then $T^{-1} : W \to V$ is linear

Theorem 2.43. *$T : V \to W$ linear* $\dim(V) < \infty$
T invertible $\iff$ $\mathrm{rk}\,(T) = \dim(V) = \dim(W)$

Proof. Use $\mathrm{rk}\,(T) + \mathrm{nul}(T) = \dim(V)$. □

Definition 2.44. $A \in M_{n\times n}(\mathbf{F})$ is invertible $:\iff$ $\exists B \in M_{n\times n}(\mathbf{F})$ with $AB = BA = Id_n$.
Then $B = A^{-1}$ inverse matrix.

(Remark: If one of $AB = Id$ or $BA = Id$, then also the other holds.)

Theorem 2.45. *$T : V \to W$ invertible, $\quad \beta, \gamma$ OB of V, W*
then $[T^{-1}]^{\beta}_{\gamma} = \Big([T]^{\gamma}_{\beta}\Big)^{-1}$ ↑
matrix inverse
(note: $[T]^{\gamma}_{\beta}$ is square mtx)

Proof. $I_n = [Id_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]^{\beta}_{\gamma}[T]^{\gamma}_{\beta}$. □

Definition 2.46. $F : V \to W$ linear bijective is called (linear) isomorphism. And V, W are isomorphic $V \simeq W$.

Theorem 2.47. *V, W finite dimensional (f.d.) VS $/\mathbf{F}$* $\quad V \simeq W \iff \dim(V) = \dim(W)$.

Corollary 2.48. *V f.d. VS $/\mathbf{F}$* $\qquad \mathbf{F}^n \simeq V \qquad n = \dim V$.

Example 2.49. $\mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4 \qquad f \overset{F}{\longmapsto} (f(1), f(2), f(3), f(4))$
dim of $\mathcal{P}_3(\mathbb{R})$, $\mathbb{R}^4$ same, F is linear, injective:
if $f(1) = f(2) = f(3) = f(4) = 0$ then
$(x-1)(x-2)(x-3)(x-4) \mid f$, but if $\deg f \le 3$, then
$f \equiv 0 \Longrightarrow f$ isomorphism.

How do we realize the isomorphism $\mathbf{F}^n \simeq V$ in corollary 2.48?

Definition 2.50. Let V be VS $/\mathbf{F}$, $\quad \dim V = n, \quad \beta$ OB.
Then $\phi_\beta : V \to \mathbf{F}^n$ given by $\phi_\beta(\mathbf{x}) = [\mathbf{x}]_\beta$ is standard representation of V w.r.t. β.

Theorem 2.51. *ϕ_β is an isomorphism.*

$\Longrightarrow \mathbf{F}^n \simeq V$ depends on the choice of basis (<u>not</u> <u>canonical</u>)

Theorem 2.52. *Let V, W VS $/\mathbf{F}$ $\quad \beta, \gamma$ OB. ($|\gamma| = m$, $|\beta| = n$).* (with $\dim V = n$, $\dim W = m$)
The map $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(\mathbf{F})$ given by $\Phi(T) = [T]_\beta^\gamma$ is an isomorphism.

Corollary 2.53. $\dim \mathcal{L}(V, W) = mn$.

2.5. <u>change of coordinates.</u>

What happens to $[\mathbf{v}]_\beta$ and $[T]_\beta^\gamma$ when changing β to β'?

Theorem 2.54. Let β, β' be OB of f.d. vector space V, let $Q = [I_V]_{\beta'}^{\beta}$. Then

(a) Q is invertible

(b) For any $\mathbf{v} \in V$, $\quad [\mathbf{v}]_\beta = Q \cdot ([\mathbf{v}]_{\beta'})$

Proof. (a) $[I_V]_\beta^{\beta'} [I_V]_{\beta'}^{\beta} = [I_V]_{\beta'}^{\beta'} = Id$ ☐

(b) $[\mathbf{v}]_\beta = [I_V(\mathbf{v})]_\beta = [I_V]_{\beta'}^{\beta} [\mathbf{v}]_{\beta'} = Q \cdot [\mathbf{v}]_{\beta'}$ ☐

Theorem 2.55. *Under the assumption of theorem 2.54 and for $T \in \mathcal{L}(V)$, we have*
$[T]_{\beta'} = Q^{-1}[T]_\beta Q$

change of basis corresponds to conjugating the coordinate matrix

Proof. $Q[T]_{\beta'} = [I_V]_{\beta'}^{\beta} [T]_{\beta'}^{\beta'}$
$= [I_V \cdot T]_{\beta'}^{\beta} = [T]_{\beta'}^{\beta}$
$\qquad \qquad ||$
$[T]_\beta Q = [T]_\beta^\beta [I_V]_{\beta'}^{\beta} = [T \cdot I_V]_{\beta'}^{\beta} = [T]_{\beta'}^{\beta}$ ☐

Definition 2.56. $(G, \cdot)$ is group $\quad g^{-1}bg$ is g-conjugate of b.

$\{ T \in \mathcal{L}(V) : T \text{ invertible} \}$ is a group, also $\{ M \in M_{n\times n}(\mathbf{F}) : M \textit{ invertible} \}$

↑
how do I see this?
determinants;
later

thus we can speak of conjugate matrix

(in book $Q^{-1}AQ$ are called <u>similar</u>; not conjugate)