3. Elementary matrix operations and systems of linear equations

goal: computing rank of matrix
solving systems of linear equations

3.1. Elementary matrix operations and elementary matrices.

goal: transform a matrix by elementary operations
into a simpler one of same rank

Definition 3.1. An elementary row-column operation is

(1) interchanging two rows/columns
(2) multiplying a row /column by a non-zero scalar
(3) adding a scalar multiple of a row /column to another row /column

Ex $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}$

need later ↓

type 1 : interchange rows 1 and 2 $\begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{pmatrix} = A_1$

(Rem If $P \to Q$ by elementary operation, then $Q \to P$ by el. oper.)

type 2 : multiply second column by 3 $\begin{pmatrix} 1 & 6 & 3 & 4 \\ 2 & 3 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix} = A_2$

type 3 : add 4× third row to first row $\begin{pmatrix} 17 & 2 & 7 & 12 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix} = A_3$

Definition 3.2. An $n \times n$ elementary matrix is one obtained from I_n by performing an elementary operation.

$$\begin{tikzcd} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ exch. row 1 and 2} \\ \text{Ex. } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \arrow[ur] \arrow[dr] & \\ & \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ add } -2\cdot \text{ row 3 to row 1} \end{tikzcd}$$

Theorem 3.3. *Let* $A \in M_{m\times n}(\mathbf{F})$ $A \longrightarrow B$ *by elementary* $\overset{[col.]}{\textit{row}}$ *operation.*
$\Longrightarrow \exists$ *elementary matrix* E *with* $B = EA$ $[B = AE]$.
with $Id \to E$ *by the same elementary operation.*

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \qquad A_2 = A \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

Theorem 3.4. *Elementary matrices are invertible, and their inverses are of the same type.*

Proof. $$\begin{pmatrix} 1 &&&&& \\ & \ddots &&&& \\ && 1 &&& \\ &&& \lambda && \\ &&&& 1 & \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 &&&&& \\ & \ddots &&&& \\ && 1 &&& \\ &&& 1/\lambda && \\ &&&& 1 & \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix} \qquad \begin{pmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& 0 && 1 && \\ &&&& \overset{1}{\ddots}\,1 &&& \\ &&& 1 && 0 && \\ &&&&&& 1 & \\ &&&&&&& \ddots \; 1 \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} 1 && \\ & \ddots & \lambda \\ && 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 && \\ & \ddots & -\lambda \\ && 1 \end{pmatrix}.$$

□

3.2. **Rank of matrix and matrix inverses.**

Definition 3.5. $A \in M_{m\times n}(\mathbf{F}) \qquad \operatorname{rk}(A) := \operatorname{rk}(L_A : \mathbf{F}^n \to \mathbf{F}^m)$.

Theorem 3.6. *Let* $\underset{\substack{\cup| \\ \beta}}{V}, \underset{\substack{\cup| \\ \gamma}}{W}$ *be VS* $/\mathbf{F}$ $\quad T : V \to W$ *linear transformation.* $\underset{}{OB}$

Then $\operatorname{rk}(T) = \operatorname{rk}\left([T]_\beta^\gamma\right)$.

Proof. The following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ \mathbf{F}^n \arrow[r, "L_{[T]_\beta^\gamma}"'] & \mathbf{F}^m \end{tikzcd}$$

$\phi_\beta(\mathbf{x}) \;=\; [\mathbf{x}]_\beta$
$\phi_\beta,\, \phi_\gamma$ isomorphism

$$\operatorname{rk}(T) \underset{\substack{\uparrow \\ \text{def of rank}}}{=} \dim(\operatorname{Im}T) \underset{\substack{\uparrow \\ \phi_\gamma \text{ iso}}}{=} \dim(\phi_\gamma(\operatorname{Im}T)) \underset{\substack{\uparrow \\ (1)}}{=} \dim(\operatorname{Im}(L_{[T]_\beta^\gamma})) \overset{\substack{\text{def of rank} \\ \downarrow}}{=} \operatorname{rk}\left(L_{[T]_\beta^\gamma}\right) \overset{\substack{\text{previous def} \\ \downarrow}}{=:} \operatorname{rk}\left([T]_\beta^\gamma\right)$$

using

$$\phi_\gamma(\operatorname{Im}T) \;=\; \operatorname{Im}(\phi_\gamma \circ T) \underset{\substack{\uparrow \\ \text{diagram commutes}}}{=} \operatorname{Im}(L_{[T]_\beta^\gamma} \circ \phi_\beta) \;=\; L_{[T]_\beta^\gamma}(\operatorname{Im}\phi_\beta) \underset{\substack{\uparrow \\ \phi_\beta \text{ iso}}}{=} L_{[T]_\beta^\gamma}(\mathbf{F}^n) \;=\; \operatorname{Im}(L_{[T]_\beta^\gamma}) \tag{1}$$

Theorem 3.7. *Let* $A \in M_{m\times n}(\mathbf{F}) \qquad \underset{\substack{\Cap \\ M_{m\times m}}}{P}, \quad \underset{\substack{\Cap \\ M_{n\times n}}}{Q}$ *invertible.*

Then $\quad \operatorname{rk}(PA) = \operatorname{rk}(A) = \operatorname{rk}(AQ) = \operatorname{rk}(PAQ)$.

Proof. $$R(L_{AQ}) = R(L_A \circ L_Q) = L_A L_Q(\mathbf{F}^n) = L_A(L_Q(\mathbf{F}^n)) \overset{\substack{L_Q \text{ onto} \\ \downarrow}}{=} L_A(\mathbf{F}^n) = \\ = R(L_A)$$

$$\implies \operatorname{rk}(AQ) = \dim(R(L_{AQ})) = \dim(R(L_A)) = \operatorname{rk}(A)\,.$$

$$\dim \underset{\substack{\| \\ \operatorname{rk}(PA)}}{R(L_{PA})} \;=\; \dim L_P(L_A(\mathbf{F}^n)) \underset{\substack{\uparrow \\ L_P \text{ iso}}}{=} \dim(L_A(\mathbf{F}^n)) \;=\; \dim R(L_A) \;=\; \operatorname{rk}(A).$$

Corollary 3.8. *Elementary row and column operations on a matrix are rank-preserving.*

Theorem 3.9. *The rank of a matrix is the maximal number of linearly independent columns* $=$ $\dim$(*subspace generated by columns*).

Proof. Let $A \in M_{m\times n}(\mathbf{F})$. Then

$$L_A : \mathbf{F}^n \to \mathbf{F}^m \qquad L_A(\mathbf{x}) = A\cdot\mathbf{x}$$

$$L_A(\mathbf{e}_i) = A\cdot\mathbf{e}_i \quad i\text{-th column of } A.$$

$$\text{So } \operatorname{Im}(L_A) = L_A(\operatorname{span}(\{\mathbf{e}_i\})) \overset{\overset{L_A \text{ linear}}{\downarrow}}{=} \operatorname{span}(\{L_A(\mathbf{e}_i)\}) = \underline{\operatorname{span}(\text{columns of } A)}. \quad \square$$

Example 3.10.

$$\operatorname{rk}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = 2$$

first two cols are linearly independent,

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Corollary 3.11. $A \in M_{m\times n}(\mathbf{F})$ *is invertible* $\iff \operatorname{rk}(A) = m = n$.

Theorem 3.12. *(important!)*

$$A \in M_{m\times n}(\mathbf{F}).\ \textit{Then } A \xrightarrow[\textit{el. op.}]{} {\scriptstyle m-r}\Big\{ \underset{I_{r,m,n}}{\underset{\shortparallel}{\left(\begin{array}{c|c} Id_r & \overbrace{0}^{n-r} \\ \hline 0 & 0 \end{array}\right)}}. \Rightarrow \underline{r = \operatorname{rk}(A)}.$$

Proof. By example.

$$\begin{pmatrix} \boxed{0} & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$$

⓪ $p :=$ pivot element row $:=$ pivot element col $:= 1$

if all elements in pivot rectangle are $0 \Rightarrow$ end $\left.\begin{matrix}\text{row}\\\text{col}\end{matrix}\right\} \geq p$

$$\begin{pmatrix} \boxed{4} & 4 & 4 & 8 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$$

① exchange rows and columns so that pivot $\neq 0$ (type 1)

$$\begin{array}{r} \\ \\ \cdot -8 \rightarrow \\ \cdot -6 \rightarrow \end{array}\begin{pmatrix} \boxed{1} & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$$

② divide row by pivot element so it gets $= 1$ (type 2)

$$\begin{array}{c} \begin{array}{ccccc} & \overset{-1}{\downarrow} & \overset{-1}{\downarrow} & \overset{-2}{\downarrow} & \end{array} \\ \left(\begin{array}{ccccc} 1 & \multicolumn{4}{c}{\boxed{1 \quad 1 \quad 2 \quad 0}} \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{array}\right) \end{array}$$

③ zero out below pivot by type 3 row operations

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{2} & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$$

④ zero out right of pivot by type 3 col operations (make ▭ entries to 0)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$$

⑤ move pivot one row down & one col right
→ ①

$$\Big\downarrow \text{②}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 & 1 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{\textcircled{\scriptsize 3-5}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{4} & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{\text{②}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \boxed{4} & 0 & 8 \\ & & 2 & 0 & 4 \end{pmatrix} \xrightarrow{\textcircled{\scriptsize 3-5}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & 0 & 0 \\ & & 0 & \boxed{0} & 0 \end{pmatrix}$$

$\xrightarrow{\text{⓪}}$ $I_{3,4,5}$ done □

Corollary 3.13. $A \in M_{m\times n}(\mathbf{F})$ $\operatorname{rk}(A) = r$ *then*
$\exists$ $\underset{\substack{\rotatebox{90}{$\in$} \\ M_{m\times m}}}{P}$, $\underset{\substack{\rotatebox{90}{$\in$} \\ M_{n\times n}}}{Q}$ *products of elementry matricies:*

$$A = P I_{r,m,n} Q\,.$$

Proof. $\underset{Q^{-1}}{P^{-1}} = E_1 \cdot \cdots \cdot E_k$ product of elementary matrices E_i for $\underset{\text{col}}{\text{row}}$ operations in ① – ⑤. Then $P = E_k^{-1} \cdot \cdots \cdot E_1^{-1}$, E_i^{-1} are elementary. □

Corollary 3.14. *Let* $A \in M_{m\times n}(\mathbf{F})$.

(a) $\operatorname{rk}(A^T) = \operatorname{rk}(A)$
(b) $\operatorname{rk}(A^T) = \dim \operatorname{span}($*columns of* $A) = \dim \operatorname{span}($*rows of* $A)$

Corollary 3.15. $A \in M_{n\times n}(\mathbf{F})$ *is invertible* $\iff$ A *is a product of elementary matrices*

Proof. "$\Longrightarrow$" A invertible $\Longrightarrow \operatorname{rk}(A) = n \overset{cor.\ 3.13}{\Longrightarrow} A = P \overset{\overset{I_{n,n,n}}{\|}}{I_n} Q = PQ$,
P, Q products of elementary matrices

"$\Longleftarrow$" elementary matrices are invertible $\Rightarrow$ their products are invertible □

Corollary 3.16. *Let* $A \in M_{m\times n}(\mathbf{F})$

$$\operatorname{rk}(A) = r \iff \exists \ \overset{\overset{M_{m\times m}}{\cup}}{P} \ , \ \overset{\overset{M_{n\times n}}{\cup}}{Q} \ \textit{invertible with } A = PI_{r,m,n}Q.$$

Proof. "$\Longrightarrow$" By Corollary 3.13, $\exists P, Q$ products of elementary matrices.
elementary matrices are invertible $\Rightarrow$ P, Q invertible

"$\Longleftarrow$" assume $A = PI_{r,m,n}Q$, $\quad P, Q$ invertible
by Corollary 3.15, P, Q are products of elementary matrices
and $\operatorname{rk}(EA) = \operatorname{rk}(A) = \operatorname{rk}(AE')$ when E, E' elementary
so $\operatorname{rk}(A) = \operatorname{rk}(PI_{r,m,n}Q) = \operatorname{rk}(I_{r,m,n}) = r$. ☐

Proof of Cor 3.14. (a) When P is invertible, $(P^T)^{-1} = (P^{-1})^T$,

$$\begin{array}{lcl} \text{so} \quad A = PI_{r,m,n}Q & \Longleftarrow & \operatorname{rk}(A) = r \\ \qquad\qquad \Downarrow & & \\ A^T = \underset{\text{invertible}}{\underset{\uparrow}{Q^T}} \ I_{r,n,m} \ \underset{\text{invertible}}{\underset{\uparrow}{P^T}} & \Longrightarrow & \operatorname{rk}(A^T) = r\,. \end{array}$$

(b) follows from (a). ☐

Theorem 3.17. *Let* $T : V \to W$, $U : W \to Z$ *linear transformations,*
A, B *matrices such that* AB *defined.*

Then (a) $\operatorname{rk}(UT) \le \begin{array}{l}\operatorname{rk}(U)\\ \operatorname{rk}(T)\end{array}$ (b) $\operatorname{rk}(AB) \le \begin{array}{l}\operatorname{rk}(A)\\ \operatorname{rk}(B)\end{array}$

Proof. (a)

$$\operatorname{rk}(UT) = \overset{\overset{(*)}{\shortparallel}}{\dim(U(T(V)))} \overset{\overset{\overset{T(V)\subseteq W}{U(T(V))\subseteq U(W)}}{\downarrow}}{\le} \dim(U(W)) = \dim\operatorname{Im}(U) = \operatorname{rk}(U).$$

$U : W \to Z$. Let $\tilde{U} = U\Big|_{T(V)} : T(V) \to Z$

$$\dim\operatorname{Im}(\tilde{U}) + \underbrace{\dim(\ker(\tilde{U}))}_{\ge 0} = \dim(T(V))$$

$$\dim\operatorname{Im}(\tilde{U}) \le \dim(T(V))$$

$$(*) = \dim(\tilde{U}(T(V))) \le \dim(T(V)) = \dim\operatorname{Im}(T) = \operatorname{rk}(T)$$

(b) for matrices apply to L_A, L_B. ☐

The inverse of a matrix

Definition 3.18. $A \in M_{m\times n}$, $\quad B \in M_{m\times p}$, $\quad \underset{\text{augmented matrix}}{\overset{\overset{(A|B)}{\shortparallel}}{(AB)}} \in M_{m\times(n+p)} \qquad (\,A\,|\,B\,)$

Let $A \in M_{n\times n}(\mathbf{F})$ consider $(A|I_n)$.
Assume A is invertible. Then A^{-1} is product of elementary matrices
(Cor 3.15 above)
$A^{-1} = E_1 \cdot \ldots \cdot E_g$.

$(A|B) \longrightarrow E(A|B) = (EA|EB)$ for an elementary matrix .
E is an elementary row operation

Now do this for $E_g, E_{g-1}, \dots, E_1$ on $(A|Id_n)$.

$(A|Id_n) \longrightarrow A^{-1}(A|Id_n) = (A^{-1}A|A^{-1}Id_n) = (Id|A^{-1})$.

Corollary 3.19. *A is invertible $\iff$ there exists a sequence of row operations that turn $(A|Id_n)$ into $(Id_n|B)$. In this case $B = A^{-1}$.*

Proof. "$\Longrightarrow$" we proved that $(A|Id_n) \to (Id_n|A^{-1})$ by row operations.
Now assume it is possible.

$$(A|Id_n) \to (Id_n|X) \qquad \text{by row operations} \tag{2}$$

row operations on $(A|X)$ preserve $X^{-1}A$: $\boxed{\text{“invariant”}}$

$$\begin{array}{cc} (EA|EX) & (A|X) \\ \downarrow & \downarrow \end{array}$$
$$(X^{-1}E^{-1})(EA) = X^{-1}A.$$

Thus if (2), then $X^{-1}Id_n = Id_n^{-1}A \quad \Rightarrow \; A = X^{-1}$
$\Rightarrow \; X = A^{-1}$

"$\Longleftarrow$" If $(A|Id_n) \to (Id_n|B)$ by row operations, then
forget right block: $A \to Id_n$ by row operations,
so $\operatorname{rk}(A) = \operatorname{rk}(Id_n) = n \Rightarrow A$ is invertible. ☐

[this means: if A is non-invertible, any attempt to turn $(A|Id_n) \xrightarrow[\text{row}]{} (Id_n|B)$ will produce a left block with a zero row/column.]

How to find row operations for $(A|Id_n) \to (Id_n|B)$?
(this is the first method to compute $B = A^{-1}$; we may see another)

<u>Ex</u> $\begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}^{-1} = \,?$ (We start as in the algorithm to determine rank.)

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} \boxed{0} & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} \boxed{2} & 4 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

① exch row 1 and 2 $\Rightarrow$ pivot$\neq 0$
if pivot + all below pivot=0
then stop. column:=0 $\Rightarrow$
<u>matrix not invertible</u>

$$\cdot -3 \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1/2 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

② multiply first row by 1/2 $\Longrightarrow$ pivot=1

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1/2 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 & -3/2 & 1 \end{array}\right)$$

③ zero out below pivot & <u>above</u> pivot (if pivot is in column > 1)

comes first important difference to row-column reduction:

cannot zero out right of pivot by column operation!

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1/2 & 0 \\ 0 & \boxed{2} & 4 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 & -3/2 & 1 \end{array}\right)$$

④ change shape of pivot rectangle (now what is <u>above</u> pivot is also in rectangle)

① unnecessary

② divide by 2 $\Rightarrow$ pivot=1

$$\begin{array}{l} \cdot -2 \\ \cdot 3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1/2 & 0 \\ 0 & \boxed{1} & 2 & 1/2 & 0 & 0 \\ 0 & -3 & -2 & 0 & -3/2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 1/2 & 0 \\ 0 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & 0 & \boxed{4} & 3/2 & -3/2 & 1 \end{array}\right)$$

③ zero <u>above</u> and below pivot

④ change pivot

$$\begin{array}{l} \cdot 3 \\ \cdot -2 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 1/2 & 0 \\ 0 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 3/8 & -3/8 & 1/4 \end{array}\right)$$

① skip

②

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/8 & -5/8 & 3/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/2 \\ 0 & 0 & 1 & 3/8 & -3/8 & 1/4 \end{array}\right)$$

③

$$\Longrightarrow \text{done} \quad \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{pmatrix}.$$

3.3. <u>**Systems of linear equations - Theoretical aspects.**</u>

$$\begin{array}{ccccc} a_{11}x_1+ & \dots & +a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1+ & \dots & +a_{mn}x_n & = & b_m \end{array} \qquad (3)$$

a_{ij}, $b_i \in \mathbf{F}$ field const

$x_j \in \mathbf{F}$ unknowns

<u>system of m linear equations with n unknowns over $\mathbf{F}$</u>

↑

we had this before!
given $\mathbf{v}_1, \dots, \mathbf{v}_n$, $\mathbf{x}$,
find λ_i with $\sum \lambda_i \mathbf{v}_i = \mathbf{x}$.
Now we do this more systematically

$A = (a_{ij})_{i=1,j=1}^{m\ \ n}$ <u>coefficient matrix</u>.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

(3) $\iff A\mathbf{x} = \mathbf{b}$. <u>linear equation system (LES)</u>

$\mathbf{s} \in \mathbf{F}^n$ <u>solution</u> $\iff$ $A\mathbf{s} = \mathbf{b}$ $\quad$ $\{$solutions$\}$ $=:$ <u>solution set</u>

$$(3) \begin{cases} \underline{\text{consistent}} : \iff \text{solution set} \neq \varnothing \\ \underline{\text{inconsistent}} : \iff \text{solution set} = \varnothing \end{cases}$$

$x_1 + x_2 = 3$	$2x_1 + 3x_2 + x_3 = 1$	$x_1 + x_2 = 0$
$x_1 - x_2 = -1$	$x_1 - x_2 + 2x_3 = 6$	$x_1 + x_2 = 1$
$x_1 = 1 \quad x_2 = 2$ unique sol	$(x_1, x_2, x_3) = \begin{cases} (-6,2,7) \\ (8,-4,-3) \end{cases}$ sol.	no sol. <u>inconsistent</u>

How do we find solution?

Definition 3.20. A system $A\mathbf{x} = \mathbf{b}$ is $\begin{cases} \underline{\text{homogeneous}} & \text{if } \mathbf{b} = \mathbf{0} \\ \underline{\text{nonhomogeneous}} & \text{if } \mathbf{b} \neq \mathbf{0} \end{cases}$

Theorem 3.21. *K sol. set of $A\mathbf{x} = \mathbf{0}$.* $\quad K = \ker(L_A) =: \ker A.$

$\dim K = n - \operatorname{rk}(L_A)$

$L_A : \mathbf{F}^n \to \mathbf{F}^m$

Proof. $\dim \ker(L_A) = \dim \mathbf{F}^n - \dim \operatorname{Im}(L_A)$. ☐

fewer eqn than variables

↓

Corollary 3.22. *If $m < n$, then $K \neq \{\mathbf{0}\}$.*

Theorem 3.23. *Let $A\mathbf{x} = \mathbf{b}$ (4) be a (consistent) LES and $A\mathbf{x} = \mathbf{0}$ (5) be the homogeneous system corresponding to (4). Then*

$$K_{(4)} = \mathbf{s} + K_{(5)} = \{\mathbf{s} + \mathbf{k} : \mathbf{k} \in K_{(5)}\},$$

where $\mathbf{s}$ is a concrete solution of (4).

Proof. Let $\mathbf{s} \in K_{(4)}$ $\quad A\mathbf{s} = \mathbf{b}$

$\mathbf{s}' \in K_{(4)}$ $\quad A\mathbf{s}' = \mathbf{b}$ $\quad A(\mathbf{s} - \mathbf{s}') = \mathbf{0} \Rightarrow \mathbf{s}' - \mathbf{s} \in K_{(5)}$

$$\Rightarrow \mathbf{s}' = (\underbrace{-\mathbf{s} + \mathbf{s}'}_{\in K_{(5)}}) + \mathbf{s} \quad \in \mathbf{s} + K_{(5)}.$$

$\Rightarrow K_{(4)} \subseteq \mathbf{s} + K_{(5)}$.

Now let $\mathbf{s}' \in \mathbf{s} + K_{(5)} \qquad \mathbf{s}' = \mathbf{s} + \mathbf{k} \qquad A\mathbf{k} = \mathbf{0}$

$$A\mathbf{s}' = A\mathbf{s} + A\mathbf{k} = A\mathbf{s} = \mathbf{b} \Rightarrow \mathbf{s}' \in K_{(4)} \Rightarrow s + K_{(5)} \subseteq K_{(4)}. \quad \square$$

Example 3.24.

$$\boxed{1} \qquad \begin{array}{rcl} -x_1 + 2x_2 - x_3 &=& 0 \\ x_1 + \ x_2 - x_3 &=& 0 \end{array} \qquad \underset{: \ \mathbb{R}^3 \to \mathbb{R}^2}{A = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 1 & -1 \end{pmatrix}} \qquad \mathrm{rk}\,(A) = 2$$

$$\dim(\ker A) = \dim(\mathbb{R}^3) - \mathrm{rk}\,(A) \ = \ 1\,.$$

$$\text{check that } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \ker A \qquad \ker A = \left\{ t \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \ : \ t \in \mathbb{R} \right\}$$

↑

($\boxed{\text{how to find – later}}$)

↓

$$\boxed{2} \qquad \begin{array}{rcl} -x_1 + 2x_2 - x_3 &=& -3 \\ x_1 + \ x_2 - x_3 &=& -2 \end{array} \qquad \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \quad \text{special solution}$$

$$\Rightarrow K = \left\{ \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \ : \ t \in \mathbb{R} \right\} \qquad \begin{array}{c} \text{solution} \\ \text{set} \end{array}$$

<u>One special case</u>

Theorem 3.25. *Let $A\mathbf{x} = \mathbf{b}$ be a LES with $A \in M_{n \times n}$.*
Then $A\mathbf{x} = \mathbf{b}$ has exactly one solution $\iff$ A is invertible
In that case the solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof. "$\Longleftarrow$" Let A be invertible $\qquad \mathbf{x} = A^{-1}\mathbf{b}$ is a solution.

$$\{\,\text{sol. of } A\mathbf{x} = \mathbf{b}\,\} \ = \ A^{-1}\mathbf{b} + \underbrace{\{\,\text{sol of } A\mathbf{x} = \mathbf{0}\,\}}_{\substack{\ker A = \{0\} \\ \uparrow \\ A \text{ invertible} \Rightarrow \text{ injective}}} \ = \ \{\,A^{-1}\mathbf{b}\,\}$$

"$\Longrightarrow$" Let $\mathbf{s}$ be the unique solution of $A\mathbf{x} = \mathbf{b}$.

$$\{\,\text{solutions to } A\mathbf{x} = \mathbf{b}\,\} \ = \ \mathbf{s} + \underbrace{\{\,\text{sol of } A\mathbf{x} = \mathbf{0}\,\}}_{\ker A} \overset{\substack{\mathbf{s} \text{ unique} \\ \downarrow}}{=} \{\,\mathbf{s}\,\}\,.$$

$\Rightarrow \ker(A) = \{\mathbf{0}\}$

$\Rightarrow A$ injective $\Rightarrow A \ : \mathbf{F}^n \to \mathbf{F}^n$ bijective $\Rightarrow A$ invertible $\quad \square$

<u>Ex.</u>

$$\begin{array}{rcl} 2x_2 \ +4x_3 &=& 3 \\ 2x_1 + 4x_2 + 2x_3 &=& 3 \\ 3x_1 + 3x_2 + x_3 &=& 1 \end{array} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\mathbf{b} = \begin{pmatrix} 1/8 & -5/8 & 3/4 \\ -1/4 & 3/4 & -1/2 \\ 3/8 & -3/8 & 1/4 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 1 \\ 1/4 \end{pmatrix}.$$

Definition 3.26. $(\,A\,|\,\mathbf{b}\,)$ the augmented matrix of LES $A\mathbf{x} = \mathbf{b}$.

Theorem 3.27. *$A\mathbf{x}=\mathbf{b}$ is consistent LES* $\iff$ $\mathrm{rk}\,(A|\mathbf{b}) = \mathrm{rk}\,(A)$.

$$\exists\mathbf{x} : A\mathbf{x}=\mathbf{b} \iff \mathbf{b}\in \underset{\substack{\shortparallel \\ (\mathrm{Im}L_A)}}{\mathrm{Im}(A)} = \text{span(columns of } A \text{)}$$

$$\iff \overbrace{\text{span(columns of } A \cup \{\mathbf{b}\}\text{)}}^{V} = \overbrace{\text{span(columns of } A\text{)}}^{W}$$

$$\left[\begin{array}{l}\text{if } V \supseteq W\text{, then}\\ V=W \iff \dim V=\dim W\end{array}\right] \longrightarrow \iff \underset{\substack{\shortparallel \\ \mathrm{rk}\,(A)}}{\dim(\text{span(col of } A\text{))}} = \underset{\substack{\shortparallel \\ \mathrm{rk}\,(A|\mathbf{b})}}{\dim(\text{span(col of } A \cup \{\mathbf{b}\}\text{))}}$$

$$\iff \mathrm{rk}\,(A) = \mathrm{rk}\,(A|\mathbf{b}) \quad \square$$

Example 3.28.
$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 0 \end{aligned} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (A\,|\,\mathbf{b}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$\mathrm{rk}\,(A) = 1 \qquad \mathrm{rk}\,(A\,|\,\mathbf{b}) = 2$$

$\Rightarrow$ inconsistent

Example 3.29. Is $\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$?

$$\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 3 \\ x_1 &= 2 \end{aligned} \qquad \text{no solution} \Rightarrow \underline{\text{no}}$$

<u>closed economic model (Application)</u>

producer : Farmer (food), Tailor (cloth), carpenter (house)
& consumer

no capital enters and exits (closed model)

	Food	Clothing	Housing	
Farmer	0.4	0.2	0.2	
Tailor	0.1	0.7	0.2	consumption
Carpenter	0.5	0.1	0.6	
(	1.0	1.0	1.0)	

e.g. Farmer consumes 40% of food and 20% of cloth.

<u>Qu</u> How much must F,T,C produce to attain an equilibrium (income = spending $\Longrightarrow$ society survives)?

Let $\overset{\text{food}}{p_1}$, $\overset{\text{cloth}}{p_2}$, $\overset{\text{house}}{p_3}$ be incomes of farmer/tailor/carpenter

when farmer must buy 20% of housing,
he spends $0.2p_3$

$$\begin{array}{rcl} \overbrace{0.4p_1 + 0.2p_2 + 0.2p_3}^{\text{spending}} & = & \overbrace{p_1}^{\text{income}} \\ 0.1p_1 + 0.7p_2 + 0.2p_3 & = & p_2 \\ 0.5p_1 + 0.1p_2 + 0.6p_3 & = & p_3 \end{array}
\qquad
\begin{array}{c} \underline{\text{consumption matrix}} \\ \downarrow \end{array}$$

$$A = \begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{pmatrix}$$

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \qquad A\mathbf{p} = \mathbf{p}.$$

In this example $\mathbf{p} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.4 \end{pmatrix}$ $\qquad \mathbf{p} \in \ker(A - Id) \iff$ equilibrium condition

so income of farmer : taylor : carpenter $= 5:7:8$

Rem. what we want is $\mathbf{p} \geq \mathbf{0}$ (positive; all entries $p_i \geq 0$)

Thm. (B.Noble 1971) If

$$A = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\quad}^{n-1} & \overbrace{\quad}^{1} \end{array} \\ \left(\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right) \end{array} \begin{array}{l} \}\, n-1 \\ \;\; 1 \end{array}$$

with $C, D > \mathbf{0}$, then

$(I - A)\mathbf{x} = \mathbf{0}$ has a 1-dimensional dolsution set
generated by a non-negative vector.

open model outside demand

d_1	food	
d_2	cloth	$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$
d_3	house	

$$\begin{array}{lcl} \text{income(carpenter)} & = & \text{spending(carpenter)} + \overbrace{\text{outside demand (house)}}^{\text{win(carpenter)}} \\ \quad\text{farmer} & & \quad\text{farmer} \qquad\qquad\qquad\qquad \text{food} \\ \quad\text{taylor} & & \quad\text{taylor} \qquad\qquad\qquad\qquad \text{cloth} \end{array}$$

$$\Downarrow$$

$\mathbf{d} + A\mathbf{p} = \mathbf{p} \Longrightarrow \mathbf{p} = (I - A)^{-1}\mathbf{d}$ (and again $\mathbf{p}$ must be non-negative)

3.4. LES - computational aspects.

we know: to solve $A\mathbf{x} = \mathbf{b}$ we need
/ 1. one sol
\ 2. basis of sol. set of $A\mathbf{x} = \mathbf{0}$

we use row operations to accomplish 1 & 2 by transforming $A\mathbf{x} = \mathbf{b}$ into
a simple system where we see solutions

Definition 3.30. $(A|\mathbf{b}) \iff (A'|\mathbf{b}')$ equivalent if $\mathbf{x}$ sol. of $A\mathbf{x} = \mathbf{b} \iff \mathbf{x}$ sol. of $A'\mathbf{x} = \mathbf{b}'$.

Theorem 3.31. *$C \in M_{m\times m}$ invertible,* $\quad A \in M_{m\times n}, \quad b \in \mathbf{F}^m$

$$(CA|C\mathbf{b}) \iff (A|\mathbf{b}) \qquad \textit{(same solutions)}$$

Corollary 3.32. *Row operations on $(A|\mathbf{b})$ give equivalent system.*

Qu: How to find proper row operations? $\boxed{\text{Gaussian elimination}}$

by example

$$\left.\begin{array}{lllll} 3x_1 & +2x_2 & +3x_3 & -2x_4 & = 1 \\ x_1 & +x_2 & +x_3 & & = 3 \\ x_1 & +2x_2 & +x_3 & -x_4 & = 2 \end{array}\right\} \quad (6)$$

$\boxed{\text{I part}}$ $\boxed{\text{Forward pass}}$

$$\left(\begin{array}{cccc|c} \boxed{3} & 2 & 3 & -2 & 1 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & 1 & -1 & 2 \end{array}\right)$$

(but not down) *(but not above)*

⓪ move pivot right as long as pivot + <u>below</u> $= 0$

① make pivot to 1

can divide by 3 but better exch row 1 & 3

$$\begin{array}{r} \\ \cdot -1 \rightarrow \\ \cdot -3 \rightarrow \end{array}\left(\begin{array}{cccc|c} \boxed{1} & 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ 3 & 2 & 3 & -2 & 1 \end{array}\right)$$

② zero out below pivot

$$\begin{array}{r} \\ \cdot -1 \\ \\ \end{array}\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & \boxed{-1} & 0 & 1 & 1 \\ 0 & -4 & 0 & 1 & -5 \end{array}\right)$$

③ move pivot 1 right & 1 down

⓪ skip

① make pivot $= 1$ ($\cdot -1$ second row)

$$\begin{array}{r} \\ \cdot 4 \\ \rightarrow \end{array}\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & -4 & 0 & 1 & -5 \end{array}\right)$$

⓪ pivot + below is zero $\longrightarrow$ move right

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & \boxed{-3} & -9 \end{array}\right)$$

①

$$\left(\begin{array}{cccc|c} \textcircled{1} & 2 & 1 & -1 & 2 \\ 0 & \textcircled{1} & 0 & -1 & -1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array}\right)$$

we achieved now that the first non-zero entry in each row = 1 & always right of first entry in the previous row

$\boxed{\text{II part}}$ $\boxed{\text{Backward pass}}$

zero out above ① using row operations of type 3 from right to left

(so that in columns with ① this is the only non-zero entry)

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array}\right) \begin{array}{l} \leftarrow \\ \leftarrow \rbrack + \\ \rbrack \end{array} +$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 5 \\ 0 & \textcircled{1} & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} \leftarrow \\ \rbrack -2 \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \quad \Longleftrightarrow \quad (7) \quad \left| \begin{array}{ccccc} x_1 & +x_3 & & = & 1 \\ x_2 & & & = & 2 \\ & & x_4 & = & 3 \end{array}\right.$$

Definition 3.33. A matrix is in <u>reduced row echelon form</u> (r.r.e. form, RREF) if the following conditions are satisfied:

(a) any non-zero row lies above any zero row

(b) first (=leftmost) non-zero entry in a row is the only non-zero entry in its column

(c) first (=leftmost) non-zero entry in a row is $= 1$ and occurs in a column (above) right to leftmost entry in previous row

What is the solution of a LES in r.r.e. form?

- for every variable that <u>does not occur</u> leftmost in an equation take arbitrary value;

  in our example $x_3 = t$
- then solve in terms of t

  $x_1 = 1 - x_3 = 1 - t \quad x_3 = t$

  $x_2 = 2 \qquad\qquad\quad x_4 = 3$

$$\Longrightarrow \quad \begin{array}{c} \text{solution} \\ \text{set of (7)} \\ \| \\ \text{solution} \\ \text{set of (6)} \end{array} = \left\{ \begin{pmatrix} 1-t \\ 2 \\ t \\ 3 \end{pmatrix} \right\}_{t \in \mathbb{R}} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Example 3.34.

$$\begin{array}{cccccc} x_1 & & & & x_5 & \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} \textcircled{1} & 0 & 2 & 0 & -2 & 3 \\ 0 & \textcircled{1} & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & \textcircled{1} & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{ccccccc} x_1 & & +2x_3 & & -2x_5 & = & 3 \\ & x_2 & -x_3 & & +x_5 & = & 1 \\ & & & x_4 & -2x_5 & = & 2 \end{array}$$

entries that do not occur first in a row: x_3 and x_5

$x_3 = s, \quad x_5 = t$

and solve for x_1, x_2, x_4

$$\begin{aligned} x_4 &= 2 + 2x_5 \;=\; 2 + 2t \\ x_2 &= 1 - x_5 + x_3 \;=\; 1 - t + s \\ x_1 &= 3 + 2x_5 - 2x_3 \;=\; 3 + 2t - 2s \end{aligned} \qquad \Longrightarrow \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 + 2t - 2s \\ 1 - t + s \\ s \\ 2 + 2t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \qquad \begin{array}{l} \text{general} \\ \quad \text{solution} \\ (s, t \in \mathbb{R} \text{ arbitrary}) \end{array}$$

Theorem 3.35. $(A|b)$ *LES in r.r.e. form with* r *non-zero equations.*

(a) $\operatorname{rk}(A) = r$

(b) *the above procedure gives the general solution*

$x = x_0 + sx_1 + tx_2 + ux_3 + \cdots \qquad s, t, u, \cdots \in \mathbb{R}$

where x_0 *is one solution of non-homogeneous system &*

$sx_1 + tx_2 + \cdots$ *is the general solution of corresponding homogeneous system*

<u>if</u> $(A|b)$ *consistent*

(c) $(A|b)$ *is inconsistent* $\Longleftrightarrow$ $\exists$ *zero row in* A *with non-zero element in* $\mathbf{b}$*;*

$$\left(\begin{array}{c|c} \cdots \quad \cdots & \cdots \\ 0 \; 0 \cdots 0 & \neq 0 \\ \cdots \quad \cdots & \cdots \end{array} \right)$$

<u>Rem</u> r.r.e. form is unique. (Why?)

<u>An interpretation of r.r.e.f.</u>

Let A be a matrix and B be the r.r.e.f. of A

Let ① be the leftmost non-zero entries in each row of B.

$\#$① $=$ rk B $=$ $\#$ non-zero rows in B

$$\text{r.r.e.f.} \Longrightarrow \underset{\substack{\| \\ \vdots \\ j_k}}{\text{column}} \text{ of } \#k \; \text{①} \text{ are } \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow k\text{-th row} \quad = \mathbf{e}_k \, .$$

<u>Claim 1:</u> $j_1, \ldots, j_r$-columns of A are linearly independent

because

$$c_1 a_{j1} + \cdots + c_r a_{jr} = 0 \qquad\qquad B = \overset{\substack{\text{invertible mtx} \\ \downarrow}}{M} A$$

$$\Big\Downarrow \qquad\qquad \text{because } A \to B \text{ by row operations}$$

$$c_1 \underset{\substack{\| \\ \mathbf{e}_1}}{M a_{j1}} + \cdots + c_r \underset{\substack{\| \\ \mathbf{e}_r}}{M a_{jr}} = 0 \Longrightarrow c_i = 0$$

Let A be invertible $\qquad \mathbf{x} = A^{-1}\mathbf{b}$ is a solution.

$$\{\,\text{sol. of } A\mathbf{x} = \mathbf{b}\,\} \;=\; A^{-1}\mathbf{b} + \underbrace{\{\,\text{sol of } A\mathbf{x} = \mathbf{0}\,\}}_{\substack{\ker A = \{0\} \\ \uparrow \\ A \text{ invertible} \Rightarrow \text{ injective}}} \;=\; \{\,A^{-1}\mathbf{b}\,\}$$

Claim 2: when $\mathbf{b}_k \;=\; \begin{pmatrix} d_1 \\ \vdots \\ d_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (8) $\quad$ is te k-th column of B then

$\mathbf{a}_k \;=\; \sum\limits_{i=1}^{r} d_i \mathbf{a}_{j_i}$ is the k-th column of A

i.e., all other $\neq j_i$-th columns
of A are linear combinations of $\mathbf{a}_{j_i}$

because $\mathbf{b}_k \;=\; \sum\limits_{i=1}^{r} d_i \mathbf{e}_i$, so

$$\mathbf{a}_k \;=\; M^{-1}\mathbf{b}_k \;\overset{(8)}{=}\; M^{-1}\left(\sum_{i=1}^{r} d_i \mathbf{e}_i\right) \;\overset{\mathbf{b}_{j_i}=\mathbf{e}_i}{=}\; \sum_{i=1}^{r} d_i M^{-1}\mathbf{b}_{j_i} \;=\; \sum_{i=1}^{r} d_i \mathbf{a}_{j_i}$$

$\Longrightarrow$ all columns of A are linear combinations of $\mathbf{a}_{i_j}$

$\Longrightarrow$ $\{\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_r}\}$ are a basis of the image of A

Theorem 3.36. *Let* $A \in M_{m\times n}$ $\quad \operatorname{rk}(A) = r$ $(r > 0)$ *and let. Then*
B be r.r.e.f. of A

(a) *The number of non-zero rows in* B *is* r.

(b) $\forall i = 1, 2, \ldots, r\ \exists$ *a column* $\mathbf{b}_{j_i}$ *of* B *s.t.* $\mathbf{b}_{j_i} = \mathbf{e}_i$

(c) $\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_r}$ *(columns of* A *numbered* $j_1, \ldots, j_r$*) are linearly independent*

(d) *For each* $k = 1, 2, \ldots, n$*: column* k *of* B *is*

$$\mathbf{b}_k \;=\; d_1\mathbf{e}_1 + \ldots + d_r\mathbf{e}_r \iff \mathbf{a}_k \;=\; d_1\mathbf{a}_{j_1} + \ldots + d_r\mathbf{a}_{j_r}$$

Corollary 3.37. *r.r.e.f of* A *is unique.*

Proof. go from $k = 1$, $k = 2$, $\ldots$, $k = n$

Let l be the number of linearly independent columns $1, \ldots, k-1$.

k-th column $\mathbf{a}_k$ of A is not lin. combination of previous columns

$\iff$ k-th column of B is not linear combination of previous columns

$\overset{B\text{ r.r.e.f.}}{\iff}$ $\mathbf{b}_k = \mathbf{e}_{l+1}$

if so, $1 + l$ is uniquely determined: it is the number in order of such k

if not, $\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_l}$ is linearly independent so $\exists$ uniquely determined d_i with $\sum_{m=1}^{l} d_m \mathbf{a}_{j_m} = \mathbf{a}_k$

$$\text{then} \qquad \sum_{m=1}^{l} d_m \underset{\mathbf{e}_m}{\underset{||}{\mathbf{b}_{j_m}}} = \mathbf{b}_k = \begin{pmatrix} d_1 \\ d_2 \\ \\ d_l \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Applications: ① choose a basis from a generating set
② complete a linearly independent set to a basis

Example 3.38.

$$A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} \enclose{circle}{1} & 2 & 0 & 4 & 0 \\ 0 & 0 & \enclose{circle}{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \enclose{circle}{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{b}_1$, $\mathbf{b}_3$, $\mathbf{b}_5$ linearly independent, $\mathbf{b}_2 = 2\mathbf{b}_1$, $\mathbf{b}_4 = 4\mathbf{b}_1 - \mathbf{b}_3$

$\Longrightarrow$ $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$ linearly independent, $\mathbf{a}_2 = 2\mathbf{a}_1$, $\mathbf{a}_4 = 4\mathbf{a}_1 - \mathbf{a}_3$

$\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ basis of span$\{\mathbf{a}_1, \dots, \mathbf{a}_5\}$

① Question: find a basis of span

$$\left\{ \overbrace{\overset{*1}{\begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}}, \begin{pmatrix} 4 \\ 2 \\ 4 \\ 6 \end{pmatrix}, \overset{*2}{\begin{pmatrix} 6 \\ 3 \\ 8 \\ 7 \end{pmatrix}}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 5 \end{pmatrix}, \overset{*3}{\begin{pmatrix} 4 \\ 1 \\ 0 \\ 9 \end{pmatrix}}}^{S} \right\}$$

which is a subset of S: answer $\{*1, *2, *3\}$.

Solution: – write the vectors in S as columns of a matrix A
– find reduced r.e.f. B of A
– choose the columns of A which correspond to columns of B with ①

② Question: Complete a linearly independent set of vectors $S \subseteq V$ to a basis of V.
– write S as columns of a matrix A
– append to A on the right as columns some generating set H of A, $A' = (A|H)$
– find reduced r.e.f. B of A'
– choose the columns of A' which correspond to columns with ① of B:
$S \cup \{\dots\}$ is a basis

Example 3.39. $V = \{\, (x_1, \dots, x_5) \in \mathbb{R}^5 \ : \ x_1 + 7x_2 + 5x_3 - 4x_4 + 2x_5 = 0\}$
$S = \{(-2,0,0,-1,-1),\ (1,1,-2,-1,-1),\ (-5,1,0,1,1)\}$
can check using ① that S is linearly independent
i.e. if we write vectors in S into a matrix A
then every column in r.r.e.f. B of A has a ①:

$$\underset{A}{\begin{pmatrix} -2 & 1 & -5 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}} \longrightarrow \underset{B}{\begin{pmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

find a generating set H of $V = \{x_1 = -7x_2 - 5x_3 + 4x_4 - 2x_5\}$

$$(x_1, x_2, x_3, x_4, x_5) = (-7t_1 - 5t_2 + 4t_3 - 2t_4, t_1, t_2, t_3, t_4) =$$

$$= t_1 \underset{*1}{\underline{\begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}} + t_2 \underset{*2}{\underline{\begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}} + t_3 \underset{*3}{\underline{\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}} + t_4 \underset{*4}{\underline{\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}} \qquad H = \{*1, *2, *3, *4\}$$

$$\left(\underbrace{\begin{array}{c:c:c:} -2 & 1 & -5 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{array}}_{S} \ \underbrace{\begin{array}{cc|c|c} -7 & -5 & 4 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}}_{H}\right) = A'$$

$$\downarrow$$

$$\left(\begin{array}{ccc:cccc} \textcircled{1} & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & \textcircled{1} & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = B' \text{ r.r.e.f.}$$

<u>answer</u>: basis of $V \supseteq S$ is

$$S \cup \{(4,0,0,1,0)\}$$

summary: we can determine rank
determine inverse
solve a LES
complete a linearly independent set to a basis
choose a basis from a generating set