4. Determinants

det : { square matrices } $\to$ $\mathbf{F}$

less important in mordern & practical applications
but in theory

- new formula for solving LES
- new formula for inverse of a matrix
- test if a matrix is regular
- calculate area and orientation or parallelogram and parallelepiped

4.1. **Determinants of order 2.**

Notation:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in M_{n\times n} \qquad \det(A) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \qquad n - \text{order of determinant}$$

Definition 4.1. $A \in M_{2\times 2}(\mathbf{F})$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det(A) = ad - bc.$

Example 4.2. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} \quad A + B = \begin{pmatrix} 4 & 4 \\ 9 & 8 \end{pmatrix}$

$\det A = 1\cdot 4 - 3\cdot 2 = -2 \quad \det B = 3\cdot 4 - 6 \cdot 2 = 0 \quad \det(A+B) = -4$

determinant is not linear. but it is linear in row addition

Theorem 4.3. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{F}^2$, $\quad k \in \mathbf{F}$.

$$\det \begin{pmatrix} \mathbf{u} + k\mathbf{v} \\ \mathbf{w} \end{pmatrix} = \det \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} + k \det \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix}$$

$\uparrow$

put 2 row vectors above / below each other to get a 2×2 matrix

$$\det \begin{pmatrix} \mathbf{w} & \mathbf{u} + k\mathbf{v} \end{pmatrix} = \det \begin{pmatrix} \mathbf{w} & \mathbf{u} \end{pmatrix} + k \det \begin{pmatrix} \mathbf{w} & \mathbf{v} \end{pmatrix}$$

$\uparrow$

(put 2 col vectors beside each other)

Proof: exercise.

first application of determinant: detect invertibility

Theorem 4.4. $A = (A_{ij}) \in M_{2\times 2}$ *is invertible* $\iff \det(A) \neq 0$. *Then*

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Proof. Assume $\det(A) \neq 0$. Then

$$A \cdot \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix}$$

$$\text{similarly} \quad \begin{pmatrix} \downarrow \\ \ \end{pmatrix} \cdot A = \ \dots$$

Now assume A is invertible. $\iff \operatorname{rk}(A) = 2$

$$\cdot - \tfrac{A_{11}}{A_{21}} \quad \overset{\operatorname{rk}\, = 2}{A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}} \quad \cdot - \tfrac{A_{21}}{A_{11}} \xrightarrow{A_{11} \neq 0} \overset{\operatorname{rk}\, = 2}{\begin{pmatrix} A_{11} & A_{12} \\ 0 & \underbrace{A_{22} - \frac{A_{12}A_{21}}{A_{11}}}_{\neq 0 \iff \det(A) \neq 0} \end{pmatrix}}$$

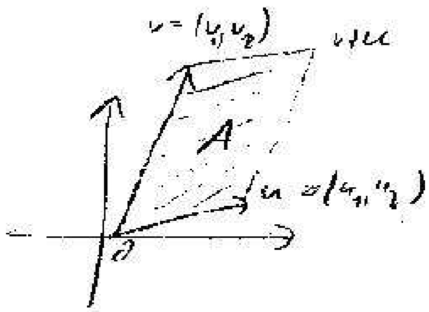
$$\Big\downarrow A_{21} \neq 0$$

$$\operatorname{rk}\, = 2 \quad \begin{pmatrix} 0 & \overbrace{A_{12} - \frac{A_{11}A_{22}}{A_{21}}}^{\neq 0 \iff \det(A) \neq 0} \\ A_{21} & A_{22} \end{pmatrix}$$

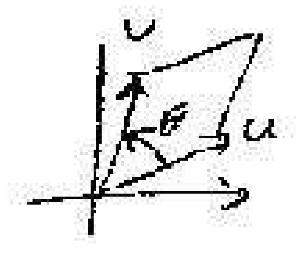
□

Area of parallelogram

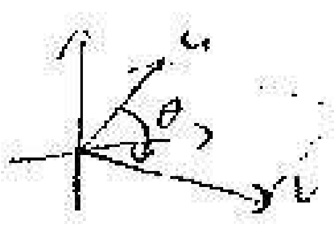
$$A = \left| \det \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} \right|$$



$\operatorname{sgn} \det \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$ is <u>orientation</u>



$\theta > 0$ positive



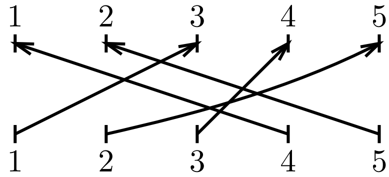
$\theta < 0$ negative

4.2. **Determinants of order n.**

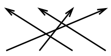
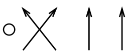
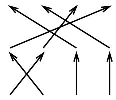
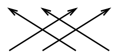
Definition 4.5. $\sigma : \overbrace{\{1,\ldots,n\}}^{\mathbb{N}_n} \to \overbrace{\{1,\ldots,n\}}^{\mathbb{N}_n}$ bijective is called *permutation* $(\sigma(1)\ \ldots\ \sigma(n))$

$$S_n := \{\sigma : \mathbb{N}_n \to \mathbb{N}_n\} \textit{ symmetric group}$$

$$(1) \quad (3\,5\,4\,1\,2) \;=\; \begin{matrix} 1 \longmapsto 3 & 3 \longmapsto 4 & 5 \longmapsto 2 \\ 2 \longmapsto 5 & 4 \longmapsto 1 & \end{matrix}$$

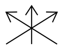


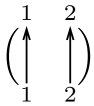
S_n is a group with multiplication given by composition $\sigma\tau := \sigma \circ \tau$.

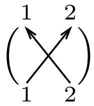
$(4\,3\,1\,2)\cdot(2\,1\,3\,4) = (4\,3\,1\,2)\circ(2\,1\,3\,4) =$  $\circ$  $=$  $=$  $= (3\,4\,1\,2)$

$|S_n| = n! = 1 \cdot 2 \cdot \ldots \cdot n.$

sign: In (1),

$(-1)^\sigma = \mathrm{sgn}(\sigma) := (-1)^{\#}$  sign of σ remark: no  !!!

example sgn  $= (-1)^0 = 1$

sgn  $= (-1)^1 = -1$

$\mathrm{sgn}\big((1)\big) = (-1)^7 = -1$

When $\mathrm{sgn}(\sigma) = 1$, we call σ *even* (or positive),
when $\mathrm{sgn}(\sigma) = -1$, we call σ *odd* (or negative).

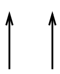
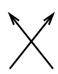
$\mathrm{sgn} : S_n \to \{-1, +1\}$ is multiplicative:

$$\mathrm{sgn}(\tau\sigma) = \mathrm{sgn}(\tau)\,\mathrm{sgn}(\sigma)\,.$$

Definition 4.6. $A = (A_{ij}) \qquad A \in M_{n\times n}$ (square matrix!)
determinant of A

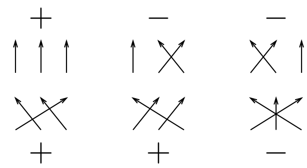
$$\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{j=1}^{n} A_{j\sigma(j)}$$

Example 4.7. $n = 2 \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

$\sigma_1 =$  $\sigma_2 =$ 

$$\det(A) = (-1)^{\sigma_1} \prod_{j=1}^{2} A_{j\sigma_1(j)} + (-1)^{\sigma_2} \prod_{j=1}^{2} A_{j\sigma_2(j)}$$

$$= \underbrace{(-1)^{\uparrow\,\uparrow}}_{1} \underset{\substack{\uparrow \\ \sigma_1(1)}}{A_{11}} \underset{\substack{\uparrow \\ \sigma_1(2)}}{A_{22}} + \underbrace{(-1)^{\times}}_{-1} \underset{\substack{\uparrow \\ \sigma_2(1)}}{A_{12}} \underset{\substack{\uparrow \\ \sigma_2(2)}}{A_{21}} = A_{11}A_{22} - A_{12}A_{21}$$

Example 4.8. $n = 3$ $\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$ $S_3 = \{$  $\}$

$$= (-1) A_{11}A_{22}A_{33} + (-1) A_{11}A_{23}A_{32} + (-1) A_{12}A_{21}A_{33}$$
$$+ (-1) A_{13}A_{21}A_{32} + (-1) A_{12}A_{23}A_{31} + (-1) A_{13}A_{22}A_{31}$$
$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}$$
$$-A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31}$$

$\Rightarrow$ gives volume of parallelepiped (평행육면체)

Why am I doing this?

Theorem 4.9. det *is linear in each row/column*

$$\det \underbrace{\begin{pmatrix} \boxed{\mathbf{a}_1} \\ \vdots \\ \boxed{\mathbf{a}_{i-1}} \\ \boxed{\mathbf{a}_i + k \cdot \mathbf{b}} \\ \boxed{\mathbf{a}_{i+1}} \\ \vdots \\ \boxed{\mathbf{a}_n} \end{pmatrix}}_{\hat{A}} = \det \underbrace{\begin{pmatrix} \boxed{\mathbf{a}_1} \\ \vdots \\ \boxed{\mathbf{a}_{i-1}} \\ \boxed{\mathbf{a}_i} \\ \boxed{\mathbf{a}_{i+1}} \\ \vdots \\ \boxed{\mathbf{a}_n} \end{pmatrix}}_{A} + k \cdot \det \underbrace{\begin{pmatrix} \boxed{\mathbf{a}_1} \\ \vdots \\ \boxed{\mathbf{a}_{i-1}} \\ \boxed{\mathbf{b}} \\ \boxed{\mathbf{a}_{i+1}} \\ \vdots \\ \boxed{\mathbf{a}_n} \end{pmatrix}}_{\tilde{A}} . \quad (2)$$

$$\det \begin{pmatrix} \boxed{\mathbf{a}_1} \cdots \boxed{\mathbf{a}_{i-1}} \boxed{\mathbf{a}_i + k \cdot \mathbf{b}} \boxed{\mathbf{a}_{i+1}} \cdots \boxed{\mathbf{a}_n} \end{pmatrix} = \det \begin{pmatrix} \boxed{\mathbf{a}_1} \cdots \boxed{\mathbf{a}_i} \cdots \boxed{\mathbf{a}_n} \end{pmatrix} + k \cdot \det \begin{pmatrix} \boxed{\mathbf{a}_1} \cdots \boxed{\mathbf{a}_{i-1}} \boxed{\mathbf{b}} \boxed{\mathbf{a}_{i+1}} \cdots \boxed{\mathbf{a}_n} \end{pmatrix} .$$

Proof.

$$\begin{aligned}
\det \hat{A} &= \sum_\sigma (-1)^\sigma \prod_{j=1}^{i-1} \hat{A}_{j\sigma(j)} \prod_{j=i} \hat{A}_{j\sigma(j)} \prod_{j=i+1}^{n} \hat{A}_{j\sigma(j)} \\
&= \sum_\sigma (-1)^\sigma \prod_{j=1}^{i-1} \hat{A}_{j\sigma(j)} \left(a_{i\sigma(i)} + kb_{\sigma(i)}\right) \prod_{j=i+1}^{n} \hat{A}_{j\sigma(j)} \\
&= \sum_\sigma (-1)^\sigma \prod_{\substack{j=1 \\ j \neq i}}^{n} \hat{A}_{j\sigma(j)} \cdot a_{i\sigma(i)} + k \sum_\sigma (-1)^\sigma \prod_{\substack{j=1 \\ j \neq i}}^{n} \hat{A}_{j\sigma(j)} \cdot b_{\sigma(i)} \\
&= \sum_\sigma (-1)^\sigma \prod_{\substack{j=1 \\ j \neq i}}^{n} A_{j\sigma(j)} \cdot A_{i\sigma(i)} + k \sum_\sigma (-1)^\sigma \prod_{\substack{j=1 \\ j \neq i}}^{n} \tilde{A}_{j\sigma(j)} \cdot \tilde{A}_{\sigma(i)} \\
&= \det A + k \cdot \det(\tilde{A}) .
\end{aligned}$$

Corollary 4.10. *If A has a zero row or column, then* $\det(A) = 0$.

Theorem 4.11. $\det \underbrace{\begin{pmatrix} \boxed{\quad \mathbf{a}_1 \quad} \\ \boxed{\quad \mathbf{a}_2 \quad} \\ \vdots \\ \end{pmatrix}}_{A} = -\det \underbrace{\begin{pmatrix} \boxed{\quad \mathbf{a}_2 \quad} \\ \boxed{\quad \mathbf{a}_1 \quad} \\ \vdots \\ \end{pmatrix}}_{\hat{A}}$ *& more generally for any 2 rows and columns*

Proof. Let $\tau = (1,2) = \begin{pmatrix} 1 \longmapsto 2 & 2 \longmapsto 1 \\ i \longmapsto i & i = 3, \dots, n \end{pmatrix} = \times \uparrow \uparrow \cdots \uparrow$. Then $\operatorname{sgn}(\tau) = -1$ and $\operatorname{sgn}(\sigma\tau) = -\operatorname{sgn}(\sigma)$. Thus multiplication by τ gives a bijective map

$$\{\,\text{odd/even permutations}\,\} \ni \sigma \longmapsto \hat{\sigma} := \sigma \circ \tau \in \{\,\text{even/odd permutations}\,\}.$$

Then

$$\begin{aligned}
\det \hat{A} &= \sum_{\sigma} (-1)^{\sigma} \prod_{j=3}^{n} a_{j\sigma(j)} \cdot a_{1\sigma(2)} \cdot a_{2\sigma(1)} \\
&= \sum_{\sigma} (-1)^{\sigma} \prod_{j=3}^{n} a_{j\hat{\sigma}(j)} \cdot a_{1\hat{\sigma}(1)} \cdot a_{2\hat{\sigma}(2)} \\
&= -\sum_{\sigma} (-1)^{\hat{\sigma}} \prod_{j=1}^{n} a_{j\hat{\sigma}(j)} \\
&\overset{\sigma \leftrightarrow \hat{\sigma}}{=} -\sum_{\hat{\sigma}} (-1)^{\hat{\sigma}} \prod_{j=1}^{n} a_{j\hat{\sigma}(j)} = -\det A\,.
\end{aligned}$$

Corollary 4.12. *If A has two equal rows or columns, then $\det(A) = 0$ (for $\operatorname{char}(\mathbf{F}) \neq 2$; otherwise exercise).*

Corollary 4.13. det is preserved under row & column operations of type III
(add to a row a multiple of another row)
det changes sign under under row & column operations of type I
(exchange 2 rows or 2 columns)
det multiplies under row & column operations of type II
(multiply row)

$$\det(I_n) = 1 \quad \left(\det = \sum_{\sigma} \prod \qquad \prod \neq 0 \text{ only if } \sigma = Id\right)$$

so can calculate determinant using row & column operations:

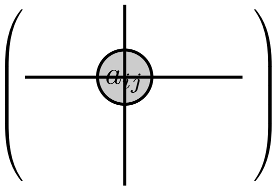
$$\begin{array}{c} \\ \cdot -4 \\ \cdot -7 \end{array} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{vmatrix} = \begin{array}{c} \begin{array}{ccc} & \cdot -2 & \cdot -3 \end{array} \\ \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -13 \end{vmatrix} \end{array} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -3 & -6 \\ 0 & -6 & -13 \end{vmatrix} = -3 \begin{array}{c} \\ \\ \cdot +6 \end{array} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -6 & -13 \end{vmatrix}$$

$$= -3\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix} = -3\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3$$

$\hookrightarrow$ complexity is n^3

Another (less practical) way : development

$\hookrightarrow$ complexity $n!$

$\widetilde{A}_{ij} =$ 

delete i-th row and j-th column

(i,j)-minor of A

row develop. $\quad \det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det \widetilde{A}_{kj} \qquad$ for every $k = 1, \ldots, n$

column develop. $\quad \det A = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det \widetilde{A}_{kj} \qquad$ for every $j = 1, \ldots, n$

Theorem 4.14. *A invertible* $\iff \det A \neq 0$.

Proof. A invertible $\longrightarrow A \longrightarrow Id$ by row and col. op.

do not change det $\neq 0$

$\Longrightarrow \det \neq 0$.

A non-inv. $\Longrightarrow A \longrightarrow$ zero row/col matrix —— " —— (동상)

$\Longrightarrow \det A = 0$.

4.3. **(More) properties of determinant.**

Theorem 4.15. $\det(A) = \det(A^T)$

Proof. $A \xrightarrow{\qquad} Id$

$A^T \xrightarrow{\qquad} Id$ $\square$

transposed

operations change determinant in same way

Theorem 4.16. *(3)* $\det(A \cdot B) = \det(A) \cdot \det(B)$.

If $\quad \det(A \cdot B) = 0 \quad \iff \quad \det(A) = 0 \quad$ or $\quad \det(B) = 0$

$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$

$A \cdot B$ not invert. $\quad \iff \quad A$ not invert. or B not invert.

(4)

$$(4) \quad \begin{cases} \text{A\& B invert.} \quad \longrightarrow & AB \text{ invert.} \\ \text{if } A \text{ not invert.} \overset{A \text{ square mx}}{\Longrightarrow} & \ker(A) \neq \{\mathbf{0}\} \\ & \Rightarrow \ \ker(BA) \neq \{\mathbf{0}\} \\ \text{if } B \text{ not invert.} \overset{B \text{ square mx}}{\Longrightarrow} & B \text{ not surj.} \\ & \Rightarrow \ BA \text{ not surj.} \end{cases}$$

so assume both hand sides are not zero in (3).

now A is invertible $\Longrightarrow A =$ product of elementary matrices

enough to prove (3) for A elementary matrix

$$A = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & 1 \ddots 1 & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \qquad \begin{array}{l} AB = B \text{ with row } i \ \& \ j \text{ exchanged. Thus} \\ \quad \det(AB) = -\det(B) = \det(A)\det(B) \Rightarrow \text{(3) ok} \end{array}$$

$$\det = -1$$

$$A = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \begin{array}{l} AB = B \text{ with row } i \text{ multiplied by } \lambda. \text{ Thus} \\ \quad \det(AB) = \lambda \det(B) = \det(A)\det(B) \qquad \text{(3) ok} \end{array}$$

$$\det = \lambda$$

$$A = \begin{pmatrix} 1 & & \\ & \ddots & \\ \alpha & & 1 \end{pmatrix} \ {}^{+\alpha} \qquad \begin{array}{l} AB = B \text{ with row } i {+}{=}\ \alpha \cdot \text{ row } j. \text{ Thus} \\ \quad \det(AB) = \det(B) = \det(A)\det(B) \qquad \text{(3) ok} \end{array}$$

$$\det = \det(Id_n) = 1$$

<u>A formula for the matrix inverse</u>

$$\boxed{(A^{-1})_{ij} = \frac{\det(\widetilde{A}_{ji}) \cdot (-1)^{i+j}}{\det(A)}}$$

Example 4.17.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{pmatrix}$$

had before
$\det A = 3$

check by development rule: $\det A = -8 \cdot 1 - (-10) \cdot 2 + 3 \cdot (-3) = 3$ ok.

$$\widetilde{A}_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 8 \end{pmatrix} \quad \widetilde{A}_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 8 \end{pmatrix} \quad \widetilde{A}_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$\det \widetilde{A}_{11} = -8 \qquad \det \widetilde{A}_{12} = -10 \qquad \det \widetilde{A}_{13} = -3$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} \det \widetilde{A}_{11} & -\det \widetilde{A}_{21} & \det \widetilde{A}_{31} \\ -\det \widetilde{A}_{12} & \det \widetilde{A}_{22} & -\det \widetilde{A}_{32} \\ \det \widetilde{A}_{13} & -\det \widetilde{A}_{23} & \det \widetilde{A}_{33} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -8 & +8 & -3 \\ +10 & -13 & +6 \\ -3 & +6 & -3 \end{pmatrix}$$

$$\det \widetilde{A}_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 8 \end{vmatrix} = -8 \qquad \det \widetilde{A}_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = -13$$

$$\det \widetilde{A}_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6$$

$$A^{-1} = \begin{pmatrix} -8/3 & 8/3 & -1 \\ 10/3 & -13/3 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

$$\det \widetilde{A}_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3 \qquad \det \widetilde{A}_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -6$$

$$\det \widetilde{A}_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3$$

This is not practical (Gauss elimination better), but has some theoretical applications.

e.g. $A \in M_{n\times n}(\mathbb{Z}) \quad \det A = \pm 1 \implies A^{-1} \in M_{n\times n}(\mathbb{Z})$

a consequence of this formula is

Theorem 4.18. *(Cramer's rule) The (unique) solution* $\mathbf{x} = A^{-1}\mathbf{b}$ *of* $A\mathbf{x} = \mathbf{b}$ *for* A *square matrix invertible*

$\mathbf{x} = (x_1, \ldots, x_n)^T$

$$x_k = \frac{\det M_k}{\det A} \qquad M_k = \textit{replace column } k \textit{ of } A \textit{ by } \mathbf{b}$$

Example 4.19.

$$\begin{aligned} x_1 \; +2x_2 \; +3x_3 &= 2 \\ x_1 \qquad\quad +x_3 &= 3 \\ x_1 \; +x_2 \; -x_3 &= 1 \end{aligned} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{\det\begin{pmatrix} \boxed{\begin{matrix}2\\3\\1\end{matrix}} & \begin{matrix}2\\0\\1\end{matrix} & \begin{matrix}3\\1\\-1\end{matrix}\end{pmatrix}}{\det(A)} = \frac{15}{6} = \frac{5}{2}$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{\det\begin{pmatrix} \begin{matrix}1\\1\\1\end{matrix} & \boxed{\begin{matrix}2\\3\\1\end{matrix}} & \begin{matrix}3\\1\\-1\end{matrix}\end{pmatrix}}{\det(A)} = \frac{-6}{6} = -1$$

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{\det\begin{pmatrix} \begin{matrix}1\\1\\1\end{matrix} & \begin{matrix}2\\0\\1\end{matrix} & \boxed{\begin{matrix}2\\3\\1\end{matrix}}\end{pmatrix}}{\det(A)} = \frac{3}{6} = \frac{1}{2} \implies \mathbf{x} = \left(\frac{5}{2}, -1, \frac{1}{2}\right)$$

(again, this is not practical for large n
as Gaussian elimination!)

but it says again: if $\det A = \pm 1$, A, $\mathbf{b}$ have integer entries, then $\mathbf{x}$ is integer solution