5. Diagonalization

plan given $T : V \to V$

Does there exist a basis β of V such that $[T]_\beta$ is diagonal
if so, how can it be found

$\longrightarrow$ eigenvalues (EV), eigenvectors, eigenspaces

5.1. Eigenvalues and eigenvectors.

$T : V \to W \qquad \beta$ OB of V, γ OB of $W \qquad \dim V = m$, $\dim W = n$

recall $\quad [T]_\beta^\gamma$ i-th column is $[T(\mathbf{v}_i)]_\gamma \qquad \beta = (\mathbf{v}_1, \dots, \mathbf{v}_m)$

$$T : V \to V \quad [T]_{\beta'} = [I_V]_\beta^{\beta'}[T]_\beta[I_V]_{\beta'}^\beta = Q^{-1}[T]_\beta Q \qquad Q = [I_V]_{\beta'}^\beta \text{ change-of-coordinate matrix}$$

Definition 5.1. $T : V \to V \qquad \dim V = n < \infty \qquad$ linear map
T is diagonalizable if $\exists$ OB β of V with
$[T]_\beta$ a diagonal matrix.

A square matrix A is diagonalizable if L_A is.
β is called diagonalizing basis.

Now if $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a diagonalizing basis with $[T]_\beta = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, then $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$, so $T(\sum a_i \mathbf{v}_i) = \sum a_i \lambda_i \mathbf{v}_i$ and $\mathbf{v}_i \neq \mathbf{0}$.

Definition 5.2. $T : V \to V$ linear operator. Assume $\mathbf{v} \neq \mathbf{0}$ and $T\mathbf{v} = \lambda \mathbf{v}$. Then we call $\mathbf{v}$ eigenvector and λ eigenvalue (EV). We say that an eigenvector corresponds to an eigenvalue, and an eigenvalue corresponds to the eigenvector.

Theorem 5.3. *$T : V \to V$ is diagonalizable $\iff$ $\exists$ basis of V of eigenvectors of T.*

Example 5.4. $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$A\mathbf{v}_1 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\mathbf{v}_1, \qquad A\mathbf{v}_2 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\mathbf{v}_2,$$

so if $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$, then $[A]_\beta (:= [L_A]_\beta) = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$

Example 5.5. $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

geomtetrically: no vector goes to a $\underset{\text{multiple}}{\overset{\text{parallel}}{\shortparallel}}$ one by $90°$ rotation
thus T has no eigenvalues / eigenvectors $\Longrightarrow$ not diagonalizable

Example 5.6. $V = C^\infty(\mathbb{R})$ $\quad C^\infty$-functions on $\mathbb{R} \to \mathbb{R}$.
$T(f) = f'$ $\qquad$ what are EV of T?

$$f' = \lambda f \Longrightarrow f = ce^{\lambda t} \neq 0$$

$\Longrightarrow$ all $\lambda \in \mathbb{R}$ are eigenvalues of T (f are *eigenfunctions*)
(for $\lambda = 0$ the eigenfunctions are the constant functions)

this cannot happen for operators on f.d. spaces

Theorem 5.7. *λ EV of $A \in M_{n\times n}(\mathbf{F})$*
$\Longleftrightarrow \det(A - \lambda Id_n) = 0$

Proof.
$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} &\Longleftrightarrow \exists \mathbf{v} \neq 0 : (A - \lambda Id_n)\mathbf{v} = 0 \\ &\Longleftrightarrow A - \lambda Id_n \text{ is not invertible} \\ &\Longleftrightarrow \det(A - \lambda Id_n) = 0 \end{aligned}$$

Definition 5.8. Let $A \in M_{n\times n}(\mathbf{F})$. $\qquad \chi_A(t) := \det(A - tId_n)$ is called characteristic polynomial of A

Example 5.9. $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

$$\begin{aligned} \det(A - tId_2) = \begin{vmatrix} 1-t & 1 \\ 4 & 1-t \end{vmatrix} &= (1-t)^2 - 4 \\ &= (t-3)(t+1) \end{aligned}$$

$\Longrightarrow$ eigenvalues of A are $+3, -1$.

Definition 5.10. Let $T : V \to V$. Let β be OB of V.
$\chi_T(t) = \det([T]_\beta - t\,Id_n)$ is called characteristic polynomial of T

Theorem 5.11. *The definition of χ_T does not depend on the choice of basis β.*

Proof. Let β, β' be OB of V. Then we know

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q \qquad\qquad Q = [I_V]^\beta_{\beta'}$$

Then

$$[T]_{\beta'} - t\,Id_n = [T - t\,Id_V]_{\beta'} = Q^{-1}[T - t\,Id_V]_\beta Q = Q^{-1}([T]_\beta - t\,Id_n)Q\,.$$

Thus

$$\begin{aligned}
\det([T]_{\beta'} - t\,Id_n) &= \overset{\displaystyle \mathbf{F}}{\overset{\cup}{\det(Q^{-1})}} \cdot \det([T]_\beta - t\,Id_n) \cdot \overset{\displaystyle \mathbf{F}}{\overset{\cup}{\det(Q)}} \\
&= \underbrace{\det(Q^{-1}) \cdot \det(Q)}_{\det(Q^{-1}\cdot Q)=\det(Id_n)=1} \cdot \det([T]_\beta - t\,Id_n) \\
&= \det([T]_\beta - t\,Id_n). \qquad \square
\end{aligned}$$

Example 5.12. $V = \mathcal{P}_2(\mathbb{R}) \quad T : V \to V \quad T(f) = f + (x+1)f'$
β SOB $\{1, x, x^2\}$

Write $[\,.\,]^\beta = ([\,.\,]_\beta)^{-1}$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \quad \Longleftarrow \quad A = [T]_\beta \qquad \begin{aligned} T(1) &= 1 \;=\; [(1,0,0)]^\beta \\ T(x) &= 2x+1 \;=\; [(1,2,0)]^\beta \\ T(x^2) &= 2x(x+1)+x^2 \;= \\ &\quad 3x^2+2x \;=\; [(0,2,3)]^\beta \end{aligned}$$

$$\begin{aligned}
\det(A - t\,Id_3) &= \det \begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} \\
&= (1-t)(2-t)(3-t)
\end{aligned}$$

$$\lambda \text{ EV} \iff \lambda = 1, 2, 3$$

Theorem 5.13. $A \in M_{n\times n}(\mathbf{F})$
$\chi_A(t) \;=\; \det(A - t\,Id_n)$ *is a polynomial in* t *of degree* n
with leading coefficient $(-1)^n$*:*

$$[\chi_A(t)]_n \;=\; (-1)^n \qquad [\chi_A(t)]_{n-1} = (-1)^{n-1}\,\mathrm{tr}A$$
$$\dots \qquad [\chi_A(t)]_0 \;=\; \det(A). \qquad \square$$

Example 5.14. (How to find eigenvectors)

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \lambda_1 = 3 \quad \lambda_2 = -1 \quad \text{(calculated before)}$$

$$B_1 = A - \lambda_1 I \;=\; \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ eigenvector to } \lambda_1 = 3 \iff \begin{aligned} -2x_1 + x_2 &= 0 \\ 4x_1 - 2x_2 &= 0 \end{aligned} \implies \mathbf{x} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (t \in \mathbb{R}\setminus\{0\})$$

$$B_2 \;=\; A - \lambda_2 I \;=\; \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ eigenvector to } \lambda_2 = -1 \iff \begin{aligned} 4x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned} \implies \mathbf{x} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (t \in \mathbb{R}\setminus\{0\})$$

eigenvector of linear operators

$$\begin{tikzcd} V \arrow[r, "T(=L_A)"] \arrow[d, "\phi_\beta"'] & V \arrow[d, "\phi_\beta"] \\ \mathbf{F}^n \arrow[r, "A\,(=[T]_\beta)"'] & \mathbf{F}^n \end{tikzcd}$$

$T : V \to V \quad \beta$ OB of $V \qquad A = [T]_\beta$

$\phi_\beta = [\,.\,]_\beta$

Lemma 5.15. $\mathbf{v}$ *is an eigenvector of* T *with EV* $\lambda \iff [\mathbf{v}]_\beta$ *is eigenvector of* A *with EV* λ.

Proof. "$\Longrightarrow$" $A[\mathbf{v}]_\beta = A\phi_\beta(\mathbf{v}) \overset{\substack{\text{diag}\\\text{commutes}\\\downarrow}}{=} \phi_\beta(T(\mathbf{v})) = \phi_\beta(\lambda\mathbf{v}) = \lambda\phi_\beta(\mathbf{v}) = \lambda[\mathbf{v}]_\beta$.

since ϕ_β is isomorphism, $\mathbf{v} \neq \mathbf{0} \Rightarrow [\mathbf{v}]_\beta = \phi_\beta(\mathbf{v}) \neq \mathbf{0}$

"$\Longleftarrow$" similar □

so, to find eigenvectors of T, we can work in any OB β.

Write $[\,.\,]^\beta = (\phi_\beta)^{-1}$. Thus $\left[\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right]^\beta = \sum_{i=1}^n a_i\mathbf{v}_i$ for $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Example 5.16. $V = \mathcal{P}_2(\mathbb{R}) \quad T(f) = f + (x+1)f' \quad \beta = \{1, x, x^2\}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad \lambda = 1, 2, 3 \text{ (calculated before)}$$

<u>Let $\lambda_1 = 1$</u> $\qquad B_1 = A - \lambda_1 Id = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \qquad \ker B_1 = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

EVec of T for EV $\lambda_1 = 1$ is $\left[t \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]^\beta = t \in \mathbb{R}$.

check: $f = t \quad T(f) = f + (x+1)f' = t + (x+1)t' = t = f$ ✔

Let $\lambda_2 = 2 \qquad B_2 = A - \lambda_2 Id = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad \ker B_2 = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

EVec of T for EV $\lambda_2 = 2$ is $\left[t \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right]^\beta = t + tx \quad (t \in \mathbb{R})$.

check: $f = t + tx \quad T(f) = f + (x+1)f' = (t+tx) + (x+1)(t+tx)'$
$= t + tx + (x+1)t = 2(t+tx) = 2f$ ✔

$$\lambda_3 = 3 \qquad B_3 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \ker B_3 = \left\{ t \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \;\; t \in \mathbb{R} \right\}$$

$$\text{r.r.e.f.} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

EVec $f = t(1 + 2x + x^2)$

check:

$$T(f) \;=\; T(t(1+2x+x^2)) = \; t(1+2x+x^2) + t(1+x)(2x+2)$$
$$t(1+x)^2 + t(x+1)(2x+2) \;=\; 3t(1+x)^2 = 3f \;✔$$

5.2. Diagonalizability.

$T : V \to V \quad \exists \beta$ with $[T]_\beta$ diagonal, find β

- test whether operator can be diagonalized
- eigenbasis to find

Theorem 5.17. *$T : V \to V$ $\lambda_1, \ldots, \lambda_k$ distinct eigenvalues with eigenvectors $\mathbf{v}_i$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ linearly independent.*

Proof. Induction over k. When $k = 1$, $\quad \{\mathbf{v}_1\}$ linearly independent $\iff$ $\mathbf{v}_1 \neq \mathbf{0}$.
Now induction step

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$ linearly independent. (1)

$$\mathbf{0} \;=\; \sum_{i=1}^{k} a_i \mathbf{v}_i \quad (2)$$

$$\mathbf{0} \;=\; (T - \lambda_k I)\mathbf{0} \;=\; (T - \lambda_k I)\Big(\sum_{i=1}^{k} a_i \mathbf{v}_i\Big) \;=\; \sum_{i=1}^{k} a_i(\lambda_i - \lambda_k)\mathbf{v}_i \;=\; \sum_{i=1}^{k-1}$$

Now by induction assumption (1), we have $a_i(\lambda_i - \lambda_k) \;=\; 0 \qquad i = 1, \ldots, k-1$
but $\lambda_i \neq \lambda_k$ by assumption $\Longrightarrow a_i = 0 \qquad i = 1, \ldots, k-1$

$$\overset{(2)}{\Longrightarrow} \mathbf{0} = a_k \mathbf{v}_k \;\underset{\mathbf{v}_k \neq \mathbf{0}}{\Longrightarrow}\; a_k \;=\; 0.$$

$\Longrightarrow$ all $a_i = 0 \quad i = 1, \ldots, k \;\Rightarrow\; \mathbf{v}_i \quad i = 1, \ldots, k$ linear independent. □

Corollary 5.18. *If $T : V \to V$ $\quad \dim V = n \quad$ If T has n distinct EV, then T diagonalizes.*

Proof. $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ eigenvectors to λ_i are linearly independent $\Rightarrow$ (eigen)basis. □

Remark 5.19. Converse is not true: Id has only one EV, but diagonalizable.

$$\mathbf{F}[t] \\ ||$$

Definition 5.20. A polynomial $f(t) \in \mathcal{P}(\mathbf{F})$ splits over $\mathbf{F}$ if $\exists\ c, \underbrace{a_1, \ldots, a_n}_{\text{(not necessarily distinct)}} \in \mathbf{F}$, $c \neq 0$ with

$$f(t) = c(t - a_1)(t - a_2) \cdot \ldots \cdot (t - a_n)$$

The algebraic multiplicity of a_i in f is $\mu_{a_i}(f) := \#\{\, j : a_j = a_i \,\}$.

Note that a_i are the roots of f $(f(a_i) = 0)$ and using factorization, one can see

every polynomial splits $\iff$ every (non-const.) polynomial has a root

Definition 5.21. $\mathbf{F}$ is algebraically closed if every polynomial in $\mathcal{P}(\mathbf{F})$ splits in $\mathbf{F}$.

Example 5.22. $f(t) = t^2 + 1 \in \mathcal{P}(\mathbb{R})$ does not split in $\mathbb{R} \Rightarrow \mathbb{R}$ is not algebraically closed

Theorem 5.23. *(Fundamental Theorem of Algebra) $\mathbb{C}$ is algebraically closed.*

Theorem 5.24. *The characteristic polynomial of any diagonalizable operator (on a f.d. VS) splits.*

Proof. $T : V \to V$

$\chi_T(t) = \chi_{[T]_\beta}(t) \qquad \forall \beta$ OB of V

so choose eigenbasis. Then $[T]_\beta$ is diagonal $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

so $\chi_{[T]_\beta}(t) = (-1)^n \prod_{i=1}^{n} (t - \lambda_i) \Rightarrow \chi_T$ splits. $\square$

Example 5.25. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \chi_A(t) = \chi_{Id}(t) = (t-1)^2 \qquad$ splits, $\lambda = 1$ only EV

If A is diagonalizable, then $[A]_\beta (= [L_A]_\beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A = Id$ ↯.

So χ_A splits, but A does not diagonalize.

Definition 5.26. $T : V \to V$ linear operator $\quad E_\lambda = \ker(T - \lambda Id) \quad \lambda$ EV is called eigenspace (of T for EV λ)

Theorem 5.27. $\dim E_\lambda \leq \mu_\lambda(\chi_T(t))$ *algebraic mult. of λ in $\chi_T(t)$.*

not always equal:

Example 5.28. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \chi_A(t) = (t-1)^2 \qquad \lambda = 1$ has algebraic multiplicity $\mu_\lambda = 2$

$\dim E_\lambda = ? \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in E_1 \;\Rightarrow\; \dim \geq 1$.

$E_\lambda \subseteq \mathbb{R}^2 \quad \dim \leq 2$. If $\dim E_\lambda = 2 \Rightarrow E_\lambda = \mathbb{R}^2$

$L_A\Big|_{E_\lambda} = \lambda Id\Big|_{E_\lambda}$ so if $E_\lambda = \mathbb{R}^2$, then

$L_A = L_A\Big|_{\mathbb{R}^2} = Id\Big|_{\mathbb{R}^2} = Id$ ↯. So $\dim E_\lambda = 1 < 2 = \mu_\lambda$.

Theorem 5.29. *Assume* $T : V \to V \quad \lambda_1, \dots, \lambda_k$ *<u>distinct EV</u>*
β_i *basis of* E_{λ_i} $\forall$ *EV* λ_i *of* T.
Then $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ *(3) is linearly independent.*

Proof. Similar to Theorem 5.17. $\square$

Theorem 5.30. $T : V \to V$ *diagonalizable* $\iff \forall \lambda_i$ *EV of* T, $i = 1, \dots, k$
$\dim E_{\lambda_i} = \mu_{\lambda_i}(\chi_T)$ *<u>and</u>*
χ_T *splits (or* $\sum_{i=1}^k \dim E_{\lambda_i} = n$*)*

Then (3) is an eigenbasis.

Proof. Theorem 5.27 + Theorem 5.29. $\square$

$\Longrightarrow$ <u>Test for diagonalization</u>

- determine characteristic polynomial of T find zeros $\Rightarrow$ eigenvalues λ_i + <u>multiplicities</u> μ_{λ_i}
- for each distinct eigenvalue λ_i, solve $(T - \lambda_i I)\mathbf{x} = 0$
  determine $m_i = \dim E_{\lambda_i} = n - \operatorname{rk}(T - \lambda_i I)$
- if for all i, $m_i = \mu_{\lambda_i}(\chi)$, then T diagonalizable, else not

Example 5.31.
$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$
linear differential equation system

(an application of diagonalization)
$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \qquad f' = \begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} \qquad f_i : \mathbb{R} \to \mathbb{R}$$

If A diagonalizes, then $\exists Q : Q^{-1}AQ = D \quad D = \operatorname{diag}(\lambda_i)$

$$Q^{-1} f' = D \cdot Q^{-1} f \qquad Q^{-1} \cdot f(t) = \left(c_i e^{\lambda_i t} \right)_{i=1}^3 \quad (c_i \in \mathbb{R})$$

$$\Longrightarrow \text{solution } f(t) = Q \cdot \left(c_i e^{\lambda_i t} \right)_{i=1}^3 .$$

(5.3 skip)

5.4. <u>Invariant subspaces and Cayley-Hamilton theorem.</u>

Definition 5.32. $T : V \to V \quad W \subset V$ is (T-)<u>invariant subspace</u> if
$T(W) \subseteq W$, i.e., $T(\mathbf{w}) \in W \ \forall \mathbf{w} \in W$.

T arbitrary
Example 5.33. $\{\mathbf{0}\}$, V, $\ker T$, $\operatorname{Im} T$, E_λ for any eigenvalue λ of T.

Example 5.34. $T : \mathbb{R}^3 \to \mathbb{R}^3 \qquad T(a,b,c) = (a+b, b+c, 0)$
$W = \{ (x,y,0) : x, y \in \mathbb{R} \}$ T-invariant

Definition 5.35. $T : V \to V \qquad \mathbf{x} \in V$
$$\Sigma_T(\mathbf{x}) := \operatorname{span}\{ \mathbf{x}, T(\mathbf{x}), T^2(\mathbf{x}), \dots \} \hookleftarrow$$
only finitely many are linearly independent

T-cyclic subspace of V generated by $\mathbf{x}$

<u>Exercise:</u> (a) $W = \Sigma_T(\mathbf{x})$ is T invariant, (b) if $\mathbf{x} \in W'$ and W' is T-invariant, then $W' \supset W$
"W is the smallest T-invariant subspace $\ni \mathbf{x}$"

Example 5.36. $T : \mathbb{R}^3 \to \mathbb{R}^3 \qquad T(a,b,c) = (-b+c, a+c, 3c)$

$$\begin{aligned} \mathbf{x} &= (1,0,0) \;=\; \mathbf{e}_1 \\ T(\mathbf{e}_1) &= (0,1,0) \;=\; \mathbf{e}_2 \\ T^2(\mathbf{e}_1) &= T(\mathbf{e}_2) \;=\; (-1,0,0) \;=\; -\mathbf{e}_1 \\ \Big[\; T^3(\mathbf{e}_1) &= -\mathbf{e}_2 \qquad T^4(\mathbf{e}_1) \;=\; \mathbf{e}_1 \Big] \quad\Longrightarrow\quad W = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \{\; (x,y,0) \;:\; x,y \in \mathbb{R} \;\} \end{aligned}$$

Example 5.37. $T : \underset{\mathbb{R}[z]}{\overset{}{\mathcal{P}(\mathbb{R})}} \to \mathcal{P}(\mathbb{R}) \qquad T(f) = f'$

$\mathcal{P}(\mathbb{R}) = \mathbb{R}[z]$

$$\mathbf{x} = z^2 \quad \Sigma_T(\mathbf{x}) \;=\; \operatorname{span}\{z^2, 2z, 2\} \;=\; \mathcal{P}_2(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$$

Theorem 5.38. $T : V \to V$ *W invariant subspace. Then*

$$\chi_{T|_W} \mid \chi_T$$

Proof. γ OB of $W \qquad \beta \supseteq \gamma$ OB of V

$$[T]_\beta \;=\; \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right) \qquad \chi_T(t) \;=\; \left|\begin{array}{c|c} B_1 - t\,Id & B_2 \\ \hline 0 & B_3 - t\,Id \end{array}\right|$$

$$= \underbrace{\det(B_1 - t\;Id)}_{\chi_{T|_W}(t)} \cdot \underbrace{\det(B_3 - t\;Id)}_{\in \mathbf{F}[t]} \qquad \square$$

Theorem 5.39. *Let* $T : V \to V \qquad W = \Sigma_T(\mathbf{v}),\; k = \dim W$. *Then*

(a) $\{\, \mathbf{v},\, T(\mathbf{v}),\, \ldots, T^{k-1}(\mathbf{v}) \,\}$ *is a basis of* W

(b) *If* $a_0\mathbf{v} + a_1T(\mathbf{v}) + \cdots + a_{k-1}T^{k-1}(\mathbf{v}) + T^k(\mathbf{v}) \;=\; \mathbf{0}$,

then $\chi_{T|_W}(t) \;=\; (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$.

Proof. (a) Let j be largest positive integer such that $\beta = \{\mathbf{v}, T(\mathbf{v}), \ldots, T^{j-1}(\mathbf{v})\}$ is linearly independent.

$\Longrightarrow T^j(\mathbf{v}) \in \operatorname{span}\beta \Longrightarrow \operatorname{span}\beta$ is T-invariant

$\xRightarrow[\text{exercise}]{} \Sigma_T(\mathbf{v}) \subseteq \operatorname{span}\beta \underset{\substack{\uparrow \\ \text{def of } \beta}}{\subseteq} \Sigma_T(\mathbf{v}) \Longrightarrow$ "$=$"

β is basis of $\Sigma_T(\mathbf{v}) \qquad |\beta| = j = \dim \Sigma_T(\mathbf{v}) \overset{\substack{\text{def of } k \\ \downarrow}}{=} k$

$\Longrightarrow j = k \Longrightarrow$ (a)

Now (b). Work in OB β

$$[T|_W]_\beta \;=\; \begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & & 0 & -a_1 \\ & {}^{1}\ddots & & \vdots \\ 0 & & 1 & -a_{k-1} \end{pmatrix} \qquad \chi_{T|_W}(t) \;=\; (-1)^k\,(\;\ldots\;). \qquad \square$$

Example 5.40. (continue example 5.36)

$T : \mathbb{R}^3 \to \mathbb{R}^3 \qquad T(a,b,c) = (-b+c, a+c, 3c)$

$$W = \Sigma_T(\mathbf{e}_1) \qquad T(\mathbf{e}_1) = \mathbf{e}_2 \quad T^2(\mathbf{e}_1) = -\mathbf{e}_1$$

$$\Longrightarrow k = 2$$

$$\Longrightarrow \boxed{1} \cdot T^2(\mathbf{e}_1) + 0 \cdot T(\mathbf{e}_1) + 1 \cdot \mathbf{e}_1 \; = \; \mathbf{0}$$

$$\overset{\text{Th 5.39}}{\Longrightarrow} \chi_{T|_W} \; = \; (-1)^2(1 + 0 \cdot t + \boxed{1} \cdot t^2) \; = \; t^2 + 1. \text{ ii}$$

check using determinant
$\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$

$$[T|_W]_\beta \; = \; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \chi_{T|_W} \; = \; \begin{vmatrix} -t & -1 \\ 1 & -t \end{vmatrix} \; = t^2 + 1$$

Definition 5.41. If $P = \sum_{i=0}^{m} a_i t^i \in \mathcal{P}(\mathbf{F})$ and $T : V \to V$, $A \in M_{n\times n}(\mathbf{F})$, then define

$$P(T) = \sum_{i=0}^{m} a_i T^i, \text{ with } T^0 = Id,\ T^1 = T, \text{ and } T^n = \underbrace{T \circ \cdots \circ T}_{n \text{ times}}$$

$$P(A) = \sum_{i=0}^{m} a_i A^i, \text{ with } A^0 = I_n,\ A^1 = A, \text{ etc. } \binom{\text{matrix}}{\text{mult}}$$

Theorem 5.42. *Let $P \in \mathcal{P}(\mathbf{F})$ and $T : V \to V$, $n = \dim V < \infty$.*

(a) *If $P(T) = 0$, then for each eigenvalue λ of T we have $P(\lambda) = 0$.*
(b) *If for each eigenvalue λ of T we have $P(\lambda) = 0$,* and T is diagonalizable, *then $P(T) = 0$.*

proof is exercise

Example 5.43. A projection T satisfies $T^2 = T$. Thus $P(T) = 0$ for $P(t) = t^2 - t$. Thus all possible eigenvalues of a projection are $\lambda = 0$ and $\lambda = 1$.

Example 5.44. A reflection T satisfies $T^2 = Id$. Thus $P(T) = 0$ for $P(t) = t^2 - 1$. Thus all possible eigenvalues of a reflection are $\lambda = 1$ and $\lambda = -1$.

Remark 5.45. It is not claimed (and not true in general) that all roots of P occur as EV of T!

Theorem 5.46. *(Cayley-Hamilton theorem)*
A linear operator $T : V \to V$ satisfies its characteristic equation

$$\chi_T(T) \; = \; 0\,.$$

Proof. Let $\mathbf{v} \neq 0$. We prove

$$\chi_T(T)(\mathbf{v}) = \mathbf{0}.$$

Let $W = \Sigma_T(\mathbf{v}) \qquad k = \dim W$

$$\left.\begin{aligned} &\overset{\text{Th 5.39(a)}}{\Longrightarrow} \quad \exists a_i \text{ with } a_0\mathbf{v} + a_1T(\mathbf{v}) + \ldots + a_{k-1}T^{k-1}(\mathbf{v}) + T^k(\mathbf{v}) = \mathbf{0} \\ &\overset{\text{Th 5.39(b)}}{\Longrightarrow} \quad \chi_{T|_W} = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k) \end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow \chi_{T|_W}(T)(\mathbf{v}) = (-1)^k(a_0Id + a_1T + \cdots + a_{k-1}T^{k-1} + T^k)(\mathbf{v}) = \mathbf{0}\,.$$

Now $\chi_{T|_W} \mid \chi_T$ by theorem 5.38, so

$\chi_T(T)(\mathbf{v}) = \mathbf{0}$. $\square$

Example 5.47. $T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \beta = (\mathbf{e}_1, \mathbf{e}_2) \quad [T]_\beta$

$$T(a,b) = (a + 2b, -2a + b) \qquad [T]_\beta = A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$\chi_A(t) = \begin{vmatrix} 1-t & 2 \\ -2 & 1-t \end{vmatrix} = (1-t)^2 + 4 = t^2 - 2t + 5$$

$$\chi_A(A) = A^2 - 2A + 5Id = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}^2 - \begin{pmatrix} 2 & 4 \\ -4 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$