6. <u>Inner Product Spaces and Normed spaces</u>

6.1. **Inner Products and Norms.**

V VS over $\mathbf{F}$ $\qquad$ $\underline{\mathbf{F} = \mathbb{R} \text{ or } \mathbb{C}}$!

Definition 6.1. A *norm* $\| \,.\, \| : V \to \mathbb{R}$ a map with the following properties:

(1) $\|\mathbf{v}\| \geqslant 0 \quad \forall \mathbf{v} \in V$
$\|\mathbf{v}\| = 0 \iff \mathbf{v} = \mathbf{0}_V$
(2) $\|c \cdot \mathbf{v}\| = |c| \cdot \|\mathbf{v}\|$ for $c \in \mathbf{F} \quad \mathbf{v} \in V$.
(3) $||\mathbf{v} + \mathbf{w}|| \leqslant ||\mathbf{v}|| + ||\mathbf{w}|| \qquad \mathbf{v}, \mathbf{w} \in V \qquad (\triangle$ inequality$)$

$(V, \| \,.\, \|)$ is called <u>normed space</u>.

Example 6.2. $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \qquad \underset{\underset{V}{\|}}{\mathbf{x} \in \mathbb{R}^n}$ standard Euclidean norm on $\mathbb{R}^n$

but there are other norms

$$\begin{aligned} \|\mathbf{x}\|_p &= \sqrt[p]{|x_1|^p + \cdots + |x_n|^p} \\ \|\mathbf{x}\|_\infty &= \max\big(|x_1|, \ldots, |x_n|\big) \end{aligned}$$

L^p spaces $(p \geqslant 1)$

$$L^p(C) = \{ f : C \to \mathbf{F} : \int_C |f|^p \, dx < \infty \}$$

$$\|f\|_p := \sqrt[p]{\int_C |f|^p \, dx}$$

$$L^\infty := \{ f \text{ bounded} \}$$

$$\|f\|_\infty := \max |f|$$

Definition 6.3. X set
a <u>distance function</u> $d : X \times X \to \mathbb{R}$
satisfies

1) $d(x,x) = 0$
$d(x,y) > 0 \quad \forall y \neq x$

2) $d(x,y) = d(y,x)$

3) $d(x,y) + d(y,z) \geqslant d(x,z) \quad \forall x, y, z \in X$

(X, d) <u>metric space</u>

Example 6.4. If $\exists \| \,.\, \|$ on V, set

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{y} - \mathbf{x}\| = \|\mathbf{x} - \mathbf{y}\|$$

$\rightsquigarrow (V, d)$ metric space

Definition 6.5. convergence $\qquad x_n, x \in X$

$$\{x_n\} \longrightarrow x \qquad \text{if } \forall \varepsilon > 0 \quad \exists N \quad \forall n \geqslant N \quad d(x_n, x) < \varepsilon$$
$$\Longrightarrow d(x_n, x_m) < 2\varepsilon$$

Definition 6.6. $\{x_n\} \subset X$ Cauchy-sequence if

$$\forall \varepsilon > 0 \quad \exists N \quad \forall n, m \geqslant N \quad d(x_n, x_m) < \varepsilon$$

$\{x_n\} \longrightarrow x$ converges $\Longrightarrow x_n$ Cauchy-sequence

Definition 6.7. (X, d) is a complete metric space if every Cauchy-sequence converges

Definition 6.8. If $(V, \|\,.\,\|)$ is a complete metric space

$$\text{with } d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|,$$

then V is called a Banach space.

Definition 6.9. an inner product $\langle\,.\,,\,.\,\rangle : V \times V \to \mathbf{F}$ satisfies the following properties:

1) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z}\rangle = \langle \mathbf{x}, \mathbf{z}\rangle + \langle \mathbf{y}, \mathbf{z}\rangle$
2) $\langle c\mathbf{x}, \mathbf{y}\rangle = c\langle \mathbf{x}, \mathbf{y}\rangle$
3) $\overline{\langle \mathbf{x}, \mathbf{y}\rangle} = \langle \mathbf{y}, \mathbf{x}\rangle$ for $\mathbf{F} = \mathbb{C}$ $\quad\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$ for $\mathbf{F} = \mathbb{R}$
4) $\langle \mathbf{x}, \mathbf{x}\rangle > 0$ if $\mathbf{x} \neq \mathbf{0}_V$ $\quad$ (3) $\Longrightarrow \langle \mathbf{x}, \mathbf{x}\rangle \in \mathbb{R}$; 2) $\Longrightarrow \langle \mathbf{0}, \mathbf{x}\rangle = 0$)

$(V, \langle\,.\,,\,.\,\rangle)$ is called inner product space (IPS).

1),2) $\quad \langle\,.\,,\,.\,\rangle$ is linear in first argument:

$$\langle \mathbf{x} + b\mathbf{y}, \mathbf{z}\rangle = \langle \mathbf{x}, \mathbf{z}\rangle + b\langle \mathbf{y}, \mathbf{z}\rangle \qquad b \in \mathbf{F}$$

3) $\Longrightarrow \langle\,.\,,\,.\,\rangle$ is conjugate linear in second argument ($\mathbf{F} = \mathbb{C}$)
linear ($\mathbf{F} = \mathbb{R}$)

$$\downarrow$$

$$\langle \mathbf{x}, \mathbf{y} + b\mathbf{z}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle + \overline{b}\langle \mathbf{x}, \mathbf{z}\rangle \qquad b \in \mathbb{C}\,.$$

linear in first & linear in second argument = bilinear
linear in first & conjugate linear in second argument = sesquilinear

Example 6.10. standard inner product on $\mathbb{C}^n$

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = \sum_{i=1}^{n} x_i \overline{y}_i$$

Example 6.11. in $\mathbb{C}^2$

$$\begin{aligned}
&\langle (1+i, 2-3i), (4, 4+5i)\rangle = \\
&(1+i)\cdot\overline{4} + (2-3i)\cdot\overline{(4+5i)} = 4 + 4i + (2-3i)(4-5i) \\
&= 4 + 4i + \big(8 + 15 - (12+10)i\big) \\
&= 27 - 18i
\end{aligned}$$

Example 6.12. $V = C([0,1]) = \{\, f : [0,1] \to \mathbb{R} \quad \text{continuous} \,\}$

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$$

Definition 6.13. $A \in M_{n(\times n)}(\mathbf{F})$ define $A^* = \overline{A^T}$ adjoint matrix

$$(A^*)_{ij} = \overline{A_{ji}}$$

Ex. $A = \begin{pmatrix} 1 & 1+2i \\ 2 & 2-3i \end{pmatrix}$ $A^* = \begin{pmatrix} 1 & 2 \\ 1-2i & 2+3i \end{pmatrix}$

Example 6.14. Frobenius inner product on $M_n(\mathbf{F}) = V$

$$\langle A, B\rangle = \operatorname{tr}(B^*A)$$

$$\langle A+B, C\rangle = \operatorname{tr}\big(C^*(A+B)\big)$$

$$= \operatorname{tr}(C^*A + C^*B) \overset{\text{tr additive; triv.}}{=} \operatorname{tr}(C^*A) + \operatorname{tr}(C^*B)$$

$$= \langle A, C\rangle + \langle B, C\rangle$$

$$\langle A, A\rangle = \operatorname{tr}(A^*A) = \sum_{i=1}^{n} (A^*A)_{ii} =$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n} (A^*)_{ik} A_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{A_{ki}} A_{ki}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \big|A_{ki}\big|^2 \iff \exists k, i : A_{k,i} \neq 0$$

$$\iff A \neq 0$$

Theorem 6.15. *If $\langle\, .\, ,\, .\,\rangle$ inner product, then* $\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$ *norm.*

moreover it setisfies the

Cauchy-Schwarz inequality $\big|\langle \mathbf{x}, \mathbf{y}\rangle\big| \leqslant \|\mathbf{x}\| \cdot \|\mathbf{y}\|$

parallelgoram identity $\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\big(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\big)$

$\rightsquigarrow$ *test normed space is not an inner product space*

Definition 6.16. If $(V, \langle\, .\, ,\, .\,\rangle)$ is a complete metric space with

$\|\,.\,\| = \sqrt{\langle\, .\, ,\, .\,\rangle}$, then it is a Hilbert space.

$$\begin{pmatrix} \text{inner product} \\ \text{inner product space} \end{pmatrix} \begin{matrix} \rightsquigarrow \\ \rightsquigarrow \end{matrix} \begin{pmatrix} \text{norm} \\ \text{normed space} \end{pmatrix} \begin{matrix} \rightsquigarrow \\ \rightsquigarrow \end{matrix} \begin{pmatrix} \text{distance} \\ \text{metric space} \end{pmatrix}$$

$\uparrow$		$\uparrow$		$\uparrow$
Hilbert space	$\rightsquigarrow$	Banach space	$\rightsquigarrow$	complete metric space
Ex. $L^2(X)$		$L^p(X)$ $p \geqslant 1$		

Theorem 6.17. *An inner product has the following properties:*

1) *sesquilinear* ($\mathbf{F} = \mathbb{C}$) *or bilinear* ($\mathbf{F} = \mathbb{R}$)
2) $\langle \mathbf{x}, \mathbf{0}\rangle = \langle \mathbf{0}, \mathbf{x}\rangle = 0$
3) $\langle \mathbf{x}, \mathbf{x}\rangle = 0 \iff \mathbf{x} = \mathbf{0}$ *(positive definiteness)*
4) *if* $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{z}\rangle$ $\forall \mathbf{x} \Rightarrow \mathbf{y} = \mathbf{z}$ *(non-degeneracy)*

Example 6.18. $V = \left\{ f : [0,2\pi] \to \mathbb{C} \quad \int_0^{2\pi} |f|^2\, dx < \infty \right\}$ $\qquad \langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\, dx$

$$= L^2([0,2\pi]) \cong L^2(S^1) \qquad \text{(Fourier calculus)}$$

Consider $f_n(t) = e^{int} \qquad n \in \mathbb{Z}$

$(f_0 \equiv 1)$

$$\begin{aligned}
\langle f_m, f_n \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imt}\, \overline{e^{int}}\, dt \qquad\qquad (m \neq n) \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t}\, dt = \frac{1}{2\pi i(m-n)} e^{i(m-n)t}\Big|_0^{2\pi} = 0 \\
\|f_n\|^2 = \langle f_n, f_n \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{int}\, \overline{e^{int}}\, dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\left|e^{int}\right|^2}_{1} dt = \frac{1}{2\pi} \cdot 2\pi = 1\,.
\end{aligned}$$

$$\langle f_m, f_n \rangle = \delta_{mn}$$

Definition 6.19. If $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, say $\mathbf{v}$ is <u>orthogonal</u> (perpendicular) to $\mathbf{w}$ and write $\mathbf{v} \perp \mathbf{w}$.
$S \subset V$ is an <u>orthogonal set</u> if $\mathbf{v} \perp \mathbf{w} \quad \forall \mathbf{v} \neq \mathbf{w} \in S$
S is <u>orthonormal</u> if S orthogonal & $\|\mathbf{v}\| = 1 \quad \forall \mathbf{v} \in S$
$S \subset V$ is a <u>complete orthonormal</u> if S orthonormal and
$\forall \mathbf{v} \in V \setminus \{\mathbf{0}\} : \exists \mathbf{v}' \in S : \langle \mathbf{v}, \mathbf{v}' \rangle \neq 0$

Remark 6.20. $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \not\ni \mathbf{0}$ orthogonal $\quad \mathbf{v}_i \rightsquigarrow \dfrac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ normalization $\to$ $S' = \left\{ \mathbf{v}_i / \|\mathbf{v}_i\| \right\}$ orthonormal set

Theorem 6.21. *(Riesz-Fischer)* $\left\{ e^{int} \right\}_{n \in \mathbb{Z}}$ *is a complete orthonormal set on* $V = L^2(S^1)$

6.2. Gram-Schmidt and orthogonal complements.

Definition 6.22. $S \subset V$ inner product space is <u>orthonormal basis</u> if S is orthonormal set and a basis

Example 6.23. $\left\{ \frac{1}{\sqrt{5}}(2,1),\ \frac{1}{\sqrt{5}}(1,-2) \right\}$ is orthonormal basis of $\mathbb{R}^2$

Example 6.24. $L^2(S^1)$ has a complete orthonormal set S which is not a basis. (It is not clear how to find an orthonormal basis.)

(by Gram-Schmidt)
$\downarrow$
We will later see that in finite dim.

complete orthonormal set $\simeq$ orthonormal basis

Theorem 6.25. *If V inner product space & $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ orthogonal set ($\mathbf{v}_i \neq \mathbf{0}$) & $\mathbf{y} \in \operatorname{span}(S)$ then*

$$\mathbf{y} \;=\; \sum_{i=1}^{k} \frac{\langle \mathbf{y}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}\, \mathbf{v}_i \tag{1}$$

in particular if S is orthonormal

$$\mathbf{y} \;=\; \sum_{i=1}^{k} \langle \mathbf{y}, \mathbf{v}_i \rangle \; \mathbf{v}_i \tag{2}$$

Proof. (2) $\Longrightarrow$ (1) by normalization, so prove (2).

Let $\mathbf{y} = \sum a_i \mathbf{v}_i \qquad \Longleftarrow \quad \mathbf{y} \in \operatorname{span}(S)$

Then

$$\langle \mathbf{y}, \mathbf{v}_j \rangle \;=\; \left\langle \sum a_i \mathbf{v}_i, \mathbf{v}_j \right\rangle \;=\; \sum a_i \langle \mathbf{v}_i, \mathbf{v}_j \rangle$$
$$=\; \sum a_i \delta_{ij} \;=\; a_j \qquad \square$$

Remark 6.26. (1) and (2) are called orthogonal decomposition of $\mathbf{y}$ (w.r.t. S)

Corollary 6.27. *V inner product space $\quad S \not\ni \mathbf{0}_V$ orthogonal ($\Longleftarrow$ orthonormal) $\Longrightarrow$ S linearly independent*

Proof. if $\sum a_i \mathbf{v}_i \;=\; \mathbf{0}_V$

$$a_j \;=\; \left\langle \sum a_i \mathbf{v}_i, \mathbf{v}_j \right\rangle \;=\; \langle \mathbf{0}_V, \mathbf{v}_j \rangle \;=\; 0 \qquad \forall j \qquad \square$$

Corollary 6.28.

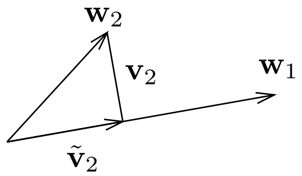
$$\|\mathbf{y}\|^2 \;=\; \sum_{i=1}^{k} \big| \langle \mathbf{y}, \mathbf{v}_i \rangle \big|^2 \qquad \{\mathbf{v}_i\} \text{ orthonormal set}$$

Proof.

$$\begin{aligned}
\langle \mathbf{y}, \mathbf{y} \rangle \;&=\; \left\langle \sum_{i=1}^{k} \overbrace{\langle \mathbf{y}, \mathbf{v}_i \rangle}^{a_i} \mathbf{v}_i, \sum_{j=1}^{k} \overbrace{\langle \mathbf{y}, \mathbf{v}_j \rangle}^{a_j} \mathbf{v}_j \right\rangle \\
&=\; \sum_{i=1}^{k} \sum_{j=1}^{k} a_i\, \overline{a_j}\, \underbrace{\langle \mathbf{v}_i, \mathbf{v}_j \rangle}_{\delta_{ij}} \\
&=\; \sum_{i=1}^{k} a_i\, \overline{a_i} \;=\; \sum_{i=1}^{k} |a_i|^2 \qquad \square
\end{aligned}$$

How to get orthonormal basis?

orthogonal projection $\quad \mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^2$ linearly independent



$$\mathbf{w}_2 - \tilde{\mathbf{v}}_2 \quad \perp \quad \tilde{\mathbf{v}}_2$$

$$\mathbf{v}_2 \;=\; \mathbf{w}_2 - c\mathbf{w}_1 \quad \perp \quad \mathbf{w}_1$$

$$\langle \mathbf{w}_2, \mathbf{w}_1 \rangle \;=\; \langle \mathbf{w}_2 - c\mathbf{w}_1, \mathbf{w}_1 \rangle + c\langle \mathbf{w}_1, \mathbf{w}_1 \rangle \;=\; c\,\|\mathbf{w}_1\|^2$$

$$\rightsquigarrow \quad c = \frac{\langle \mathbf{w}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}$$

$$\tilde{\mathbf{v}}_2 \;=\; \frac{\langle \mathbf{w}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\,\mathbf{w}_1$$ orthogonal projection of $\mathbf{w}_2$ onto $\mathbf{w}_1$

$$\mathbf{v}_2 \;=\; \mathbf{w}_2 - \tilde{\mathbf{v}}_2 \;=\; \mathbf{w}_2 - \frac{\langle \mathbf{w}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\,\mathbf{w}_1$$ orthogonalization of $\mathbf{w}_2$ w.r.t. $\mathbf{w}_1$

Theorem 6.29. *Let $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be linear independent set of inner product space V. Then*

$$\mathbf{v}_k \;=\; \frac{\mathbf{w}_k - \sum_{j=1}^{k-1} \langle \mathbf{w}_k, \mathbf{v}_j \rangle \mathbf{v}_j}{\left\| \mathbf{w}_k - \sum_{j=1}^{k-1} \langle \mathbf{w}_k, \mathbf{v}_j \rangle \mathbf{v}_j \right\|}$$

gives an orthonormal set $S' = \{\mathbf{v}_k\}$ with $\operatorname{span}(S') = \operatorname{span}(S)$

Gram-Schmidt orthonormalization

Example 6.30. $\mathbb{R}^4 \qquad \mathbf{w}_1 = (1,0,1,0) \quad \mathbf{w}_2 = (1,1,1,1) \quad \mathbf{w}_3 = (0,1,2,1)$

$$\mathbf{v}_1 \;=\; \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} \;=\; \frac{1}{\sqrt{2}}(1,0,1,0)$$

$$\mathbf{v}_2' = \mathbf{w}_2 - \langle \mathbf{w}_2, \mathbf{v}_1 \rangle \mathbf{v}_1 \;=\; (1,1,1,1) - \frac{1}{\sqrt{2}} \cdot 2 \cdot \frac{1}{\sqrt{2}}(1,0,1,0)$$

$$= \; (0,1,0,1)$$

$$\mathbf{v}_2 \;=\; \frac{(0,1,0,1)}{\|(0,1,0,1)\|} \;=\; \frac{1}{\sqrt{2}}(0,1,0,1)$$

$$\mathbf{v}_3' = \mathbf{w}_3 - \langle \mathbf{w}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{w}_3, \mathbf{v}_2 \rangle \mathbf{v}_2$$

$$= \; (0,1,2,1) - \frac{1}{\sqrt{2}} \cdot 2 \cdot \frac{1}{\sqrt{2}}(1,0,1,0) - \frac{1}{\sqrt{2}} \cdot 2 \cdot \frac{1}{\sqrt{2}}(0,1,0,1)$$

$$= \; (-1,0,1,0)$$

$$\mathbf{v}_3 \;=\; \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} \;=\; \frac{1}{\sqrt{2}}(-1,0,1,0)$$

Example 6.31. $V = \mathcal{P}(\mathbb{R}) \qquad \langle f, g\rangle = \int_{-1}^{1} f \cdot g$

$$S = \{1,\ x,\ x^2\}$$

$\tilde{\mathbf{v}}_1 = \mathbf{w}_1 = 1$

$$\|\mathbf{w}_1\|^2 = \int_{-1}^{1} 1^2\, dx = 2 \qquad \mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \underline{\underline{\frac{1}{\sqrt{2}}}}$$

$$\mathbf{w}_2 = x \qquad \tilde{\mathbf{v}}_2 = \mathbf{w}_2 - \langle \mathbf{w}_2, \mathbf{v}_1\rangle \mathbf{v}_1 = \mathbf{w}_2 = x \qquad \langle \mathbf{w}_2, \mathbf{v}_1\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} 1 \cdot x\, dx = 0$$

$$\mathbf{v}_2 = \frac{\tilde{\mathbf{v}}_2}{\|\tilde{\mathbf{v}}_2\|} = \underline{\underline{\sqrt{\frac{3}{2}}x}} \qquad \|\tilde{\mathbf{v}}_2\|^2 = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$$

$$\mathbf{w}_3 = x^2 \qquad \tilde{\mathbf{v}}_3 = \mathbf{w}_3 - \langle \mathbf{w}_3, \mathbf{v}_1\rangle \mathbf{v}_1 - \langle \mathbf{w}_3, \mathbf{v}_2\rangle \mathbf{v}_2 \qquad \langle \mathbf{w}_3, \mathbf{v}_1\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} x^2\, dx = \frac{\sqrt{2}}{3}$$

$$= \mathbf{w}_3 - \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} \qquad \langle \mathbf{w}_3, \mathbf{v}_2\rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} x^3\, dx = 0$$

$$= x^2 - \frac{1}{3} \qquad \|\tilde{\mathbf{v}}_3\|^2 = \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9}\, dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$$

$$\mathbf{v}_3 = \frac{\tilde{\mathbf{v}}_3}{\|\tilde{\mathbf{v}}_3\|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) = \underline{\underline{\sqrt{\frac{5}{8}}(3x^2 - 1)}}$$

Example 6.32. (continued)

express $f = 1 + 2x + 3x^2$ as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

$f = \sum a_i \mathbf{v}_i \qquad a_i = \langle f, \mathbf{v}_i\rangle$ calculate inner product instead of solving a linear eqn system !

$$a_1 = \int_{-1}^{1} \frac{1}{\sqrt{2}}\left(1 + 2x + 3x^2\right) dx = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x^2 + x^3 \Big|_{-1}^{1}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$a_2 = \int_{-1}^{1} \sqrt{\frac{3}{2}}\, x\left(1 + 2x + 3x^2\right) dx = \sqrt{\frac{3}{2}}\left(\frac{x^2}{2} + \frac{2}{3}x^3 + \frac{3}{4}x^4\right)\Bigg|_{-1}^{1} = \frac{4}{3}\sqrt{\frac{3}{2}} = \frac{2\sqrt{6}}{3}$$

$$a_3 = \int_{-1}^{1} \sqrt{\frac{5}{8}}\,(3x^2 - 1)\left(1 + 2x + 3x^2\right) dx = \ldots = \sqrt{\frac{8}{5}}$$

fin. dim. space $\rightsquigarrow$ ∞ dim. (Hilbert) space

$\{\mathbf{v}_i\}$ orthonormal basis

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i$$

(ortho. decomp.)

$$\|\mathbf{x}\|^2 = \sum_{i=1}^{n} \left|\langle \mathbf{x}, \mathbf{v}_i \rangle\right|^2$$

$\{\mathbf{v}_i\}$ (countable) complete orth. set

$$\mathbf{x} = \sum_{i=1}^{\infty} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i$$

converges in norm

$$\left\| \mathbf{x} - \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i \right\| \xrightarrow[n \to \infty]{} 0$$

$$\|\mathbf{x}\|^2 = \sum_{i=1}^{\infty} \left|\langle \mathbf{x}, \mathbf{v}_i \rangle\right|^2$$

When $V = L^2([0, 2\pi])$ and $S = \left\{ e^{int} \right\}$, the orthogonal decomposition

$$f(t) = \sum_{n=-\infty}^{\infty} \left\langle f, e^{int} \right\rangle e^{int}$$

is called Fourier series (and $\left\langle f, e^{int} \right\rangle$ Fourier coefficients) of f.

Example 6.33. $\underset{\substack{\Downarrow \\ f(t) = t}}{L^2([0, 2\pi])} \supset S = \{ \underbrace{e^{int}}_{f_n} : n \in \mathbb{Z} \}$

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} t \cdot \bar{t} \, dt \underset{t \in \mathbb{R}}{=} \frac{1}{2\pi} \int_0^{2\pi} t^2 \, dt = \frac{4\pi^2}{3}.$$

$$\langle f, f_0 \rangle = \frac{1}{2\pi} \int_0^{2\pi} t \cdot \bar{1} \, dt = \frac{2\pi^2}{2\pi} = \pi \qquad |\langle f, f_0 \rangle|^2 = \pi^2$$

$$\langle f, f_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} t \cdot \overline{e^{int}} \, dt = \frac{1}{2\pi} \int_0^{2\pi} t \cdot e^{-int} \, dt$$

$$= \frac{1}{2\pi} \Bigg(\underbrace{\frac{t}{-in} e^{-int} \Bigg|_0^{2\pi}}_{\frac{2\pi}{-in}} - \underbrace{\frac{1}{(-in)^2} e^{-int} \Bigg|_0^{2\pi}}_{0} \Bigg)$$

$$\int t e^{-int} \, dt = \frac{t}{-in} e^{-int} - \frac{1}{(-in)^2} e^{-int} + C$$

$$= -\frac{1}{in} = \frac{i}{n} \qquad \left(i = \sqrt{-1} \right)$$

$$|\langle f, f_n \rangle|^2 = \frac{1}{n^2} \qquad n \in \mathbb{Z} \backslash \{0\}$$

Thus

$$\frac{4\pi^2}{3} = \|f\|^2 = \sum_{n=-\infty}^{\infty} \left|\left\langle f, e^{int}\right\rangle\right|^2 = \pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}\,,$$

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$ <u>even Riemann zeta values</u>

$\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z}$ converges absolutely for $z \in \mathbb{C}$ if $\Re e\, z > 1$

Riemann zeta function

($\rightsquigarrow$ Riemann hypothesis etc.)

similarly it is known

$$\zeta(2k) = \pi^{2k}\cdot\langle\text{something rational}\rangle$$

but odd $\zeta(2k+1)$?? odd Riemann zeta values?

$\zeta(3) \notin \mathbb{Q}$ is known, and

$\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\left\{\zeta(2k+1) : k \geqslant 1\right\} = \infty$ latest cry of research !

$\downarrow$

(so ∞ly many are irrational, but which?)

Example 6.34. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+n+1}$ $\displaystyle\left(n+\frac{1}{2}\right)^2+\frac{3}{4}$

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\overline{g}$$

$$f = e^{-it/2+\frac{\sqrt{3}}{2}t} \in L^2([0,2\pi])$$

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{3}t}\,dt = \frac{1}{2\sqrt{3}\pi}\left(e^{2\sqrt{3}\pi}-1\right)$$

$$\left\langle f, e^{int}\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{t\left(\frac{\sqrt{3}}{2}+\left(-1/2-n\right)i\right)}\,dt$$

$$= \frac{1}{2\pi\left(\frac{\sqrt{3}}{2}-\left(n+1/2\right)i\right)}\left(-e^{\sqrt{3}\pi}-1\right) \qquad e^{-(2n+1)\pi i} = -1$$

$$\left|\left\langle f, e^{int}\right\rangle\right|^2 = \frac{1}{(2\pi)^2\left(\frac{3}{4}+\left(n+1/2\right)^2\right)}\left(e^{\sqrt{3}\pi}+1\right)^2$$

$$n^2+n+1$$

$$\|f\|^2 = \sum_{n=-\infty}^{\infty}\left|\left\langle f, e^{int}\right\rangle\right|^2$$

$$
\begin{aligned}
\frac{1}{2\sqrt{3}\pi}\left(e^{2\sqrt{3}\pi}-1\right) &= \sum_{n=-\infty}^{\infty} \frac{1}{(2\pi)^2(n^2+n+1)}\left(e^{\sqrt{3}\pi}+1\right)^2 \\
\frac{2\pi}{\sqrt{3}}\cdot\frac{e^{\sqrt{3}\pi}-1}{e^{\sqrt{3}\pi}+1} &= \sum_{n=-\infty}^{\infty} \frac{1}{n^2+n+1} \\
&= \sum_{n=0}^{\infty} \frac{1}{n^2+n+1} + \sum_{n=0}^{\infty} \frac{1}{(-n-1)^2+(-n-1)+1} = 2\sum_{n=0}^{\infty} \frac{1}{n^2+n+1} \\
1.798 \approx \frac{\pi}{\sqrt{3}}\,\frac{e^{\sqrt{3}\pi}-1}{e^{\sqrt{3}\pi}+1} &= \sum_{n=0}^{\infty} \frac{1}{n^2+n+1}
\end{aligned}
$$

orthogonal complements

Definition 6.35. set $S \subset V$ inner prod. space

$$S^{\perp} = \{\, \mathbf{x} \in V \,:\, \forall s \in S \quad \langle \mathbf{x}, \mathbf{s} \rangle = 0 \,\}$$

orthogonal complement of S

Theorem 6.36. *$S^{\perp}$ is a linear subspace of V.*

Example 6.37. $\{\mathbf{0}\}^{\perp} = V \qquad V^{\perp} = \{0\}$

$$\forall \mathbf{v} \in V \quad \langle \mathbf{x}, \mathbf{v} \rangle = 0 \overset{\mathbf{x}=\mathbf{v}}{\Longrightarrow} \langle \mathbf{x}, \mathbf{x} \rangle = 0$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$$

$$\Rightarrow \|\mathbf{x}\| = 0 \Rightarrow \mathbf{x} = \mathbf{0}$$

Example 6.38. Let $V = L^2([0, 2\pi]) \quad S = \operatorname{span}\{\underbrace{e^{int}}_{S'}\}$

$$S'^{\perp} = S^{\perp} = \{\mathbf{0}\} \text{ although } S \neq V$$

S' complete orth. system

so V has a proper subspace with trivial orthogonal complement (this does not happen in finite dim.)

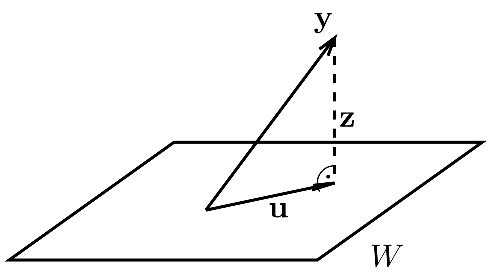
Theorem 6.39. *(orthogonal decomposition)*

Let $V \supset W$ f.d. subspace $\mathbf{y} \in V$.

Then $\mathbf{y} = \underset{W}{\underset{\Cap}{\mathbf{u}}} + \underset{W^{\perp}}{\underset{\Cap}{\mathbf{z}}}$ *unique decomposition*

and $\mathbf{u}$ 'closest' to $\mathbf{y}$ on W:

$$\|\mathbf{y} - \mathbf{u}\| \leqslant \|\mathbf{y} - \mathbf{x}\| \quad \forall \mathbf{x} \in W$$



Proof. $\mathbf{u} := \sum \langle \mathbf{y}, \mathbf{v}_i \rangle \mathbf{v}_i$ for some orthonormal basis of W

$$\mathbf{z} := \mathbf{y} - \mathbf{u}$$

$$\|\mathbf{y}-\mathbf{x}\|^2 = \|\mathbf{u}+\mathbf{z}-\mathbf{x}\|^2 = \|(\mathbf{u}-\mathbf{x})+\mathbf{z}\|^2$$

$$= \|\mathbf{u}-\mathbf{x}\|^2 + \|\mathbf{z}\|^2 \geqslant \|\mathbf{z}\|^2 = \|\mathbf{y}-\mathbf{u}\|^2 . \qquad \square$$

$\uparrow$

$\mathbf{u}-\mathbf{x} \in W$, $\mathbf{z} \in W^\perp$ $\quad \langle \mathbf{u}-\mathbf{x}, \mathbf{z}\rangle = 0$

Definition 6.40. $\mathbf{u}$ <u>orthogonal projection</u> of $\mathbf{y}$ on W
$\mathbf{z}$ <u>orthogonalization</u> of $\mathbf{y}$ w.r.t. W

Theorem 6.41. *V fin. dim. VS $\quad$ W subspace*

$$V = W \oplus W^\perp \qquad \dim(V) = \dim(W) + \dim(W^\perp)$$

6.3. <u>**The a adjoint of a linear operator.**</u>

$A \in M_n(\mathbf{F}) \quad \rightsquigarrow \quad A^* = \overline{A^T}$

now we define the corresponding linear operator

Theorem 6.42. *V fin. dim. inner product space $/\mathbf{F}$*

$$\forall g : V \to \mathbf{F} \quad \text{linear} \quad \exists!\ \mathbf{x} \in V : \forall \mathbf{y} \in V \ \ g(\mathbf{y}) = \langle \mathbf{y}, \mathbf{x}\rangle$$

I can represent a linear map to $\mathbf{F}$ as an inner product with a vector $\mathbf{x}$.

Proof. $\{\mathbf{v}_i\}$ orthonormal basis

$$\mathbf{x} = \sum \overline{g(\mathbf{v}_i)}\mathbf{v}_i$$

$\mathbf{x}$ unique because $\langle\ ,\ \rangle$ non-degenerate. $\square$

Now let

$T : V \to V \quad$ linear $\quad \mathbf{y} \in V$

$$\begin{array}{ccc} \mathbf{x} & \longmapsto & \langle T(\mathbf{x}), \mathbf{y}\rangle \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ V & & \mathbf{F} \end{array} \qquad \text{is map to } \mathbf{F} \text{ linear in } \mathbf{x}$$

$\overset{\text{Theorem}}{\rightsquigarrow} \quad \exists(!)\, \mathbf{y}' \in V \ \ \langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}'\rangle$

map $\quad \mathbf{y} \mapsto \mathbf{y}'$ is called <u>adjoint map to T</u>

Definition 6.43. $T : V \to V \qquad$ Define $T^* : V \to V$ by

$$\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T^*(\mathbf{y})\rangle \quad \forall\, \mathbf{x}, \mathbf{y} \in V$$

Theorem 6.44. *T^* is linear and if β is orthonormal basis of V then*

$$[T^*]_\beta = [T]_\beta^* \ \left(= \overline{[T]_\beta^T}\ \right)$$

(<u>Proof: book</u>)

Corollary 6.45. *$A \in M_n(\mathbf{F})$* $\qquad L_{A^*} = \big(L_A\big)^*$

$$(T^*)^* = T$$

6.4. Self-adjoint operators.

Definition 6.46.

$$\begin{array}{lll} T : V \to V & \text{self-adjoint (Hermitian)} & \text{if } T = T^* \\ A \in M_n(\mathbf{F}) & \text{self-adjoint} & \text{if } L_A = L_A^* \iff A = A^* \end{array}$$

$$\hookrightarrow \quad \begin{array}{ll} \mathbf{F} = \mathbb{R} & A = A^T \text{ symmetric matrix} \\ \mathbf{F} = \mathbb{C} & A = \overline{A}^T \ \underline{\text{Hermitian matrix}} \end{array}$$

$$\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T(\mathbf{y})\rangle \qquad \forall \mathbf{x}, \mathbf{y}.$$

Theorem 6.47. *a) λ EV $\Longrightarrow \lambda$ real   b) eigenvectors to different EVs are $\perp$*

Proof. a) $\underset{\lambda\langle \mathbf{x}, \mathbf{x}\rangle}{\underset{\parallel}{\langle \lambda\mathbf{x}, \mathbf{x}\rangle}} = \langle T(\mathbf{x}), \mathbf{x}\rangle = \langle \mathbf{x}, T(\mathbf{x})\rangle = \langle \mathbf{x}, \lambda\mathbf{x}\rangle = \overline{\lambda}\langle \mathbf{x}, \mathbf{x}\rangle$

$\langle \mathbf{x}, \mathbf{x}\rangle \neq 0 \quad \to \quad \underline{\lambda = \overline{\lambda}}$

$T(\mathbf{x}) = \lambda_1\mathbf{x}$
$T(\mathbf{y}) = \lambda_2\mathbf{y}$

b) $\langle T(\mathbf{x}), \mathbf{y}\rangle = \lambda_1 \langle \mathbf{x}, \mathbf{y}\rangle$

$\parallel$

$\langle \mathbf{x}, T(\mathbf{y})\rangle = \overline{\lambda_2}\langle \mathbf{x}, \mathbf{y}\rangle \underset{\text{a)}}{\underset{\uparrow}{=}} \lambda_2 \langle \mathbf{x}, \mathbf{y}\rangle \qquad \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0}\langle \mathbf{x}, \mathbf{y}\rangle = 0$ ☐

Theorem 6.48. *T self-adjoint $\Longrightarrow \chi_T$ splits.*

Proof. clearly over $\mathbf{F} = \mathbb{C}$. Over $\mathbf{F} = \mathbb{R}$ regard T as an operator over $\mathbb{C}$, then χ_T splits over $\mathbb{C}$, but all EV are real, thus χ_T also splits over $\mathbb{R}$. ☐

Theorem 6.49. *(Schur) T self-adjoint $\Longrightarrow$ T is diagonalizable and has an orthonormal eigenbasis*

Definition 6.50. $T : V \to V$ self-adjoint is called <u>positive/negative</u> definite
semidefinite

$$\text{if } \langle T(\mathbf{x}), \mathbf{x}\rangle \begin{array}{c} > \\ < \\ \geqslant \\ \leqslant \end{array} 0 \qquad \forall \mathbf{x} \neq \mathbf{0}_V$$

A positive/negative (semi)definite $\iff$ L_A is so

Theorem 6.51. *T, A*

positive def.	$\iff$ *all EV* λ	> 0
negative def.		$<$
positive semidef.		$\geqslant$
negative semidef.		$\leqslant$

Example 6.52. $f(x_1, \ldots, x_n) \in \mathbb{R} \qquad f : \mathbb{R}^n \to \mathbb{R}$

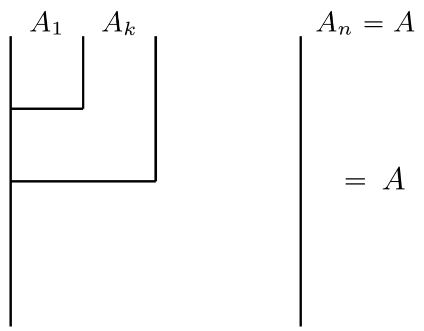
$\nabla f(x_1, \ldots, x_n) = 0$ critical point

gradient

$$\mathrm{Hess}_f(\mathbf{x}_0) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\right)_{i,j=1}^{n}$$

Theorem 6.53. $\mathbf{x}_0$ *crtitical point of* f $\quad \nabla f(\mathbf{x}_0) = 0$

$$\begin{array}{llll} \mathrm{Hess}_f(\mathbf{x}_0) & \textit{positive def.} \Longrightarrow \mathbf{x}_0 & \textit{loc. minimum} \Longrightarrow \mathrm{Hess}_f(\mathbf{x}_0) & \textit{pos. semidef.} \\ & \textit{negative} & \textit{maximum} & \textit{neg.} \end{array}$$



$$A_k = \left(a_{ij}\right)_{i,j=1}^{k}$$
principal minor

Theorem 6.54. *(principal minor test)* $\qquad A \in M_n(\mathbf{F}) \quad A = A^*$

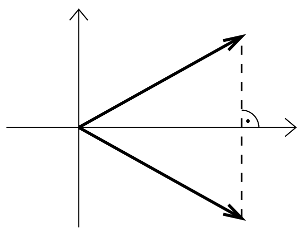
$$\left.\begin{array}{lllcrl} A & \textit{pos} & \textit{def.} & \Longleftrightarrow & \det A_k & > 0 \\ & \textit{neg} & \textit{def.} & \Longleftrightarrow & (-1)^k \det A_k & > 0 \\ & \textit{pos} & \textit{semid.} & \Longleftrightarrow & \det A_k & \geqslant 0 \\ & \textit{neg} & \textit{semid.} & \Longleftrightarrow & (-1)^k \det A_k & \geqslant 0 \end{array}\right\} \quad \forall k = 1, \ldots, n$$

6.5. Unitary and orthogonal operators and their matrices.

Definition 6.55. $T : V \to V$ is unitary ($\mathbf{F} = \mathbb{C}$) or orthogonal ($\mathbf{F} = \mathbb{R}$) if $\|T(\mathbf{x})\| = \|\mathbf{x}\| \quad \forall \mathbf{x} \in V$.
preserves length $\qquad A$ o/u $:\Longleftrightarrow$ L_A o/u

Ex. reflection in $\mathbb{R}^2$

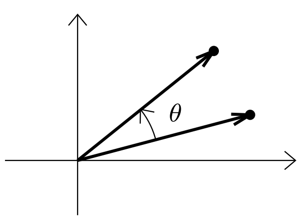
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



Ex. $\pm Id$, $\quad \lambda\, Id$
$\mathbf{F} = \mathbb{R} \qquad |\lambda| = 1$
$\mathbf{F} = \mathbb{C}$

rotation $\theta \in (0, 2\pi)$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Theorem 6.56. $\dim V < \infty$

$$\begin{aligned} T \textit{ orthogonal/untary} &\Longleftrightarrow TT^* = T^*T = Id \\ &\Longleftrightarrow T \textit{ invertible and } T^{-1} = T^* \end{aligned}$$

Remark 6.57. T o/u $\Longrightarrow$ T injective. But if $\dim V = \infty$ not always surjective. For example, on

$$V = l^2 = \{\, (x_1, \ldots, x_n, \ldots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty \,\} \quad \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

consider

$$T((x_1,\dots,x_n,\dots)) = (0,x_1,\dots,x_n,\dots))\,.$$

Lemma 6.58.
$$\begin{aligned} \langle \mathbf{x},\mathbf{y}\rangle &= \frac{\|\mathbf{x}+\mathbf{y}\|^2-\|\mathbf{x}\|^2-\|\mathbf{y}\|^2}{2} && \mathbf{F}=\mathbb{R} \\ \Re e\,\langle \mathbf{x},\mathbf{y}\rangle &= \textit{(same)} && \\ \Im m\,\langle \mathbf{x},\mathbf{y}\rangle &= \Re e\,\langle \mathbf{x},i\mathbf{y}\rangle && \end{aligned}\Bigg\}\ \mathbf{F}=\mathbb{C}$$

Lemma 6.59. *T is orthogonal/unitary $\iff$ T preserves $\langle\,.\,,\,.\,\rangle$, i.e.,*
$$\langle T(\mathbf{x}),T(\mathbf{y})\rangle = \langle \mathbf{x},\mathbf{y}\rangle \quad \forall \mathbf{x},\mathbf{y}\in V$$

Proof. "$\Longrightarrow$" e.g., $\mathbf{F}=\mathbb{C}$

$$\begin{aligned} \Im m\ \langle T(\mathbf{x}),T(\mathbf{y})\rangle &= \Re e\ \langle T(\mathbf{x}),iT(\mathbf{y})\rangle \\ &= \Re e\ \langle T(\mathbf{x}),T(i\mathbf{y})\rangle \\ &= \frac{\|T(\mathbf{x})+iT(\mathbf{y})\|^2-\|T(\mathbf{x})\|^2-\|T(i\mathbf{y})\|^2}{2} \\ &= \frac{\|T(\mathbf{x}+i\mathbf{y})\|^2-\|T(\mathbf{x})\|^2-\|T(i\mathbf{y})\|^2}{2} \\ &\overset{\substack{T\text{ unitary}\\ \downarrow}}{=} \frac{\|\mathbf{x}+i\mathbf{y}\|^2-\|\mathbf{x}\|^2-\|i\mathbf{y}\|^2}{2} \\ &= \Re e\ \langle \mathbf{x},i\mathbf{y}\rangle = \Im m\ \langle \mathbf{x},\mathbf{y}\rangle \end{aligned}$$ □

<u>Proof of theorem 6.56.</u>

$$\begin{aligned} \langle T(\mathbf{x}),T(\mathbf{y})\rangle &= \langle \mathbf{x},\mathbf{y}\rangle \quad \forall \mathbf{x},\mathbf{y}\in V \\ \| \quad & \\ \langle T^*T(\mathbf{x}),\mathbf{y}\rangle & \end{aligned}$$

$$\xRightarrow{\substack{\langle\,.\,,\,.\,\rangle\\ \text{non-degenerate}}} \quad \mathbf{x}=T^*T(\mathbf{x}) \quad \Rightarrow \quad T^*T = Id$$ □

Corollary 6.60. $|\det(T)|=1$.

Proof. $\det(T^*T) = \det(T^*)\det(T) = \overline{\det(T)}\ \det(T)$. □

Theorem 6.61. *T unitary/orth. λ EV $\Longrightarrow |\lambda|=1$.*

Proof. $\|\mathbf{x}\| = \|T(\mathbf{x})\| = \|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$. □

Example 6.62. (!!) rotation shows that T may not have EV
(over $\mathbf{F}=\mathbb{R}$) in particular not diagonalizable!

(Below we'll see this doesn't occur for $\mathbf{F}=\mathbb{C}$.)

Theorem 6.63. *T unitary/orth. eigenvectors to different EV are $\perp$.*

Proof. $T(\mathbf{x}) = \lambda_1\,\mathbf{x}$
$T(\mathbf{y}) = \lambda_2\,\mathbf{y}$

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y}\rangle &= \langle T(\mathbf{x}), T(\mathbf{y})\rangle \\ &= \langle \lambda_1\,\mathbf{x}, \lambda_2\,\mathbf{y}\rangle = \lambda_1\overline{\lambda}_2\,\langle \mathbf{x}, \mathbf{y}\rangle.\end{aligned}$$ If $\langle \mathbf{x}, \mathbf{y}\rangle \neq 0$,

then $\lambda_1\overline{\lambda}_2 = 1$, but $|\lambda_1|^2 = \lambda_1\overline{\lambda}_1 = 1$
$\implies \lambda_1 = \lambda_2.$ ∎

Example 6.64. orthogonal maps of $\mathbb{R}^2$
- rotation R_θ $\quad \det = 1$
- $R_\theta \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \det = -1$

Theorem 6.65. *T orthogonal / unitary $V \to V$, $\dim V < \infty$, $W \subset V$ subspace*

$$\big(T(W)\big)^\perp = T\big(W^\perp\big)$$

Proof. $\mathbf{x} \in \big(T(W)\big)^\perp \iff \forall \mathbf{w} \in W\ \ \langle \mathbf{x}, T(\mathbf{w})\rangle = 0$

Let $\mathbf{y} \in V$. Then $T(\mathbf{y}) = \mathbf{x} \in \big(T(W)\big)^\perp$

$$\begin{aligned} &\langle T(\mathbf{y}), T(\mathbf{w})\rangle = 0\ \ \forall \mathbf{w} \in W \\ &\qquad\quad \| \\ \iff &\langle \mathbf{y}, \mathbf{w}\rangle = 0\ \ \forall \mathbf{w} \in W \\ \iff &\mathbf{y} \in W^\perp \end{aligned}$$

$T^{-1}\Big(\big(T(W)\big)^\perp\Big) = W^\perp$ $\quad$ apply T ∎

Corollary 6.66. *$T : V \to V$ over $\mathbf{F} = \mathbb{C}$.* $\quad \dim V = n < \infty$

unitary $\implies T$ diagonalizable

Proof. Induction over $n = \dim V$.

χ_T has a root $\lambda \in \mathbb{C}$ (because $\mathbb{C}$ is algebraically closed)

Let $\mathbf{v} \in V$ be eigenvector to EV λ.

$S = \mathbf{F}\mathbf{v} = \operatorname{span}(\{\mathbf{v}\})$.

Then $T(S) = S$.
Thus $T\big(S^\perp\big) = T(S)^\perp = S^\perp$,
i.e., $S^\perp$ is an invariant subspace of V of dimension $n-1$

Consider $T' = T\Big|_{S^\perp} : S^\perp \to S^\perp$ unitary

By induction $T' = T\Big|_{S^\perp}$ is diagonalizable

$\exists \beta' \subset S^\perp$ OB

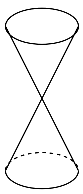
$$\beta' = \{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\} \quad [T']_{\beta'} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n-1} \end{bmatrix}.$$

Consider OB $\beta = \{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ of V.

Then $[T]_\beta = \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & [T']_{\beta'} \end{array}\right] = \left[\begin{array}{c|ccc} \lambda & & 0 & \\ \hline & \lambda_1 & & 0 \\ 0 & & \ddots & \\ & 0 & & \lambda_{n-1} \end{array}\right]$

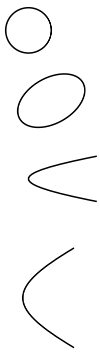
$\implies T$ diagonalizable. $\square$

7. Conics (원뿔곡선)



cone — cone $\cap$ plane (except if going through the vertex)

원뿔 is a circle, ellipse, parabola, 쌍곡선 hyperbola



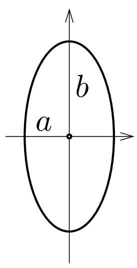
$\left.\begin{array}{l}\text{circle}\\ \text{ellipse}\\ \text{parabola}\\ \text{hyperbola}\end{array}\right\}$ conics 원뿔곡선

Example 7.1. shed light with a flashlight on a wall (all types occur)
planet orbits are ellipses
path of water in a sprinkle is a paralbola
if you sharpen a pencil, you get a hyperbola

7.1. The general conic.

all $\underset{\mathbb{R}^2}{\overset{\mathbf{x}}{\Cap}} = (x_1, x_2)$ with $x_1^2 + x_2^2 = r^2$ form a circle of radius r at $(0,0)$.

(3) $\lambda_1 x_1^2 + \lambda_2 x_2^2 \overset{\lambda_1,\lambda_2>0}{=} c$ gives an ellipse



$a = \sqrt{\dfrac{c}{\lambda_1}}$

$b = \sqrt{\dfrac{c}{\lambda_2}}$

Let us rewrite (3) as

(4) $\mathbf{x}^T D\mathbf{x} - c = 0 \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

If we rotate (around the origin) the ellipse by α,
we get that $\tilde{\mathbf{x}} = R_{-\alpha}\mathbf{x}$ must satisfy (4).

$(R_{-\alpha}^T \;=\; R_\alpha =: R)$

$$(5) \qquad \mathbf{x}^T RDR^T \mathbf{x} - c \;=\; 0$$

equation for rotated ellipse at origin

If we <u>rotate and translate</u> the ellipse by $\mathbf{v}$ then

$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{v}$ must satisfy (5).

$$(\mathbf{x}^T - \mathbf{v}^T)RDR^T(\mathbf{x} - \mathbf{v}) - c \;=\; 0$$

equation for ellipse in general position

Let $A = RDR^T$
$\quad \mathbf{b} = A\mathbf{v}, \qquad d = \mathbf{v}^T A\mathbf{v} - c$

$\mathbf{x}^T A\mathbf{x} - 2\mathbf{x}^T\mathbf{b} + d \;=\; 0$

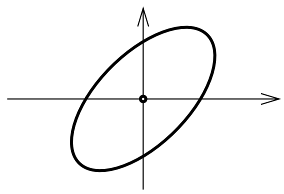
Example 7.2. Start with ellipse $2x_1^2 + 4x_2^2 = 1$

$$\mathbf{x}^T \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}}_{D} \mathbf{x} - 1 \;=\; 0 \qquad (6)$$

What equation do we get when we rotate by $\dfrac{\pi}{4} = 45^\circ$?

$$R_{\pi/4} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \quad \longleftarrow \quad \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \qquad \begin{aligned} \sin(\pi/4) &= \\ \cos(\pi/4) &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$A = RDR^T = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$



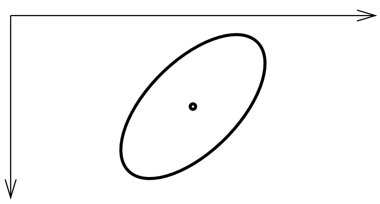
$\mathbf{x}^T A\mathbf{x} - c \;=\; 0$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \mathbf{x}^T A\mathbf{x} \;=\; 3x_1^2 + 3x_2^2 - 2x_1x_2$$

회전된
$\rightsquigarrow$ equation for rotated ellipse

$$3x_1^2 + 3x_2^2 - 2x_1x_2 - 1 \;=\; 0$$

translated 옮겨진



Now we translate by a vector $\quad \mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\mathbf{b} = A\mathbf{v} \;=\; \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \;=\; \underline{\begin{pmatrix} 7 \\ -5 \end{pmatrix}}$$

$$\begin{aligned} d = \begin{pmatrix} \underset{\substack{\uparrow \\ v_1}}{2} & \underset{\substack{\uparrow \\ v_2}}{-1} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} - 1 \;&=\; 3v_1^2 + 3v_2 + 2 - 2v_1v_2 - 1 \\ &=\; 3 \cdot 2^2 + 3 \cdot (-1)^2 - 2 \cdot 2 \cdot (-1) \\ &=\; 12 + 3 + 4 - 1 \;=\; 18 \end{aligned}$$

$\mathbf{x}^T A\mathbf{x} - 2\mathbf{x}^T\mathbf{b} + d \;=\; 0$

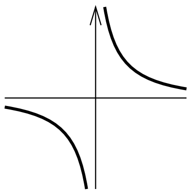
$$\mathbf{x}^T \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \mathbf{x} - 2\mathbf{x}^T \begin{pmatrix} 7 \\ -5 \end{pmatrix} + 18 \;=\; 0 \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$3x_1^2 + 3x_2^2 - 2x_1x_2 - 14x_1 + 10x_2 + 18 \;=\; 0$$

(end of Ex.)

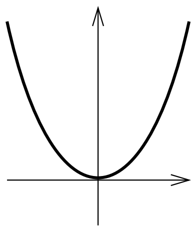
Similar forms work for every conic.

Example 7.3. hyperbola $\qquad x_2 = \dfrac{1}{x_1}$



$$x_1x_2 \;=\; 1$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x} - 1 \;=\; 0 \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



Example 7.4. parabola $\qquad x_2 = x_1^2$

$$x_1^2 - x_2 \;=\; 0$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \mathbf{x}^T \begin{pmatrix} 0 \\ -1 \end{pmatrix} \;=\; 0$$

General form of a conic

$$c_1x_1^2 + c_2x_2^2 + c_3x_1x_2 + c_4x_1 + c_5x_2 + c_6 \quad = \quad 0 \quad (7)$$

$$\Updownarrow$$

$$\mathbf{x}^T \underbrace{\begin{bmatrix} c_1 & 1/2 c_3 \\ 1/2 c_3 & c_2 \end{bmatrix}}_{A} \mathbf{x} \;-\; 2\mathbf{x}^T \begin{pmatrix} -1/2 c_4 \\ -1/2 c_5 \end{pmatrix} + \; c_6 \quad = \quad 0$$

7.2. **Analyzing conics.** Question: given (7), what type of conic is it?

first analyze A

A is diagonalizable; let λ_1, λ_2 be eigenvalues
write $|A| = \det A$.

1. if $\lambda_1, \lambda_2 < 0$ or $\lambda_1, \lambda_2 > 0$, then $\qquad$ (elliptic case)

   $\Updownarrow$

   $\lambda_1\lambda_2 = |A| > 0$

   conic is an *ellipse* or *circle*

   $\uparrow \qquad \uparrow$

   $\lambda_1 \neq \lambda_2 \quad \lambda_1 = \lambda_2 \iff A$ scalar

   or it is a degenerate case: a $\quad$ point $\quad$ or empty

   e.g. $\quad x_1^2 + x_2^2 = 0 \qquad x_1^2 + x_2^2 + 1 = 0$

2. $|A| = 0 \to$ w.l.o.g. $\lambda_1 = 0$ (parabolic case)

if $\lambda_2 = 0 \longrightarrow A = 0$ degenerate (line for $c_4 \neq 0$ or
$c_5 \neq 0$
$\mathbb{R}^2$ or $\varnothing$ for $c_4 = c_5 = 0$)

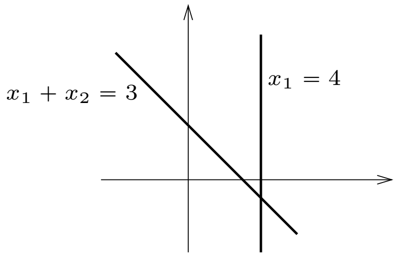
$\lambda_2 \neq 0 \longrightarrow$ conic is a parabola
or degenerate (2 parallel lines, 1 line or empty)

3. $|A| < 0$ (λ_1, λ_2 have opposite sign) (hyperbolic case)

퇴화
conic is a hyperbola or degenerate case:
2 lines

E.g. $(x_1 + x_2 - 3)(x_1 - 4) = 0$

$x_1^2 + x_1x_2 - 7x_1 - 4x_2 + 12 = 0$



How do we see if some conic is degenerate?
We will check it using the center.

7.3. **Position of a conic and degeneracy test.** Every non-parabolic conic (incl degenerate) is obtained from one in
standard form

$$\mathbf{x}^T D\mathbf{x} - c = 0 \qquad \left(\text{center } \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ (8) by $\underbrace{\text{rotation}}_{\text{B}}$ and $\underbrace{\text{translation}}_{\text{A}}$.

Write conic in form

$$\mathbf{x}^T A\mathbf{x} - 2\mathbf{x}^T\mathbf{b} + d = 0 \tag{9}$$

A translation vector $\mathbf{v}$ satisfies $A\mathbf{v} = \mathbf{b}$
you know A and $\mathbf{b}$; solve for $\mathbf{v}$. if $|A| \neq 0$

$\mathbf{v} \in \mathbb{R}^2$ is center (we regard it as a point now)

Degeneracy test for non-parabolic conics ($|A| \neq 0$)

1) if center lies on the conic, then the conic degenerates

정칙
2) Otherwise the conic is either regular, or
$|A| > 0$ and the conic is empty.

To finish degeneracy test, we must know how
to test for empty conic. We assume $|A| > 0$.

Evaluate the r.h.s. of (9) for $\mathbf{x} = \mathbf{v}$

get some number $t \in \mathbb{R}$:

$$t \;=\; \mathbf{v}^T A\mathbf{v} - 2\mathbf{v}^T\mathbf{b} + d\,.$$

(If $t = 0$, conic is a point.)

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \qquad \left.\begin{array}{l} \text{If } \lambda_1, \lambda_2 > 0 \;(\iff a > 0) \text{ and } t > 0, \text{ or} \\ \lambda_1, \lambda_2 < 0 \;(\iff a < 0) \text{ and } t < 0 \end{array}\right\} \iff a \cdot t > 0$$

then conic is empty, otherwise it is a circle or ellipse (regular).

In that case, up to translation and rotation we have the basic form (3) with $c = -t$, thus

$$a = \sqrt{-\frac{t}{\lambda_1}}, \quad b = \sqrt{-\frac{t}{\lambda_2}}, \quad \text{area} = \pi \cdot a \cdot b = \frac{|t|\pi}{\sqrt{\lambda_1\lambda_2}} = \frac{|t|\pi}{\sqrt{|A|}}$$

Example 7.5. $x_1^2 + x_1x_2 - 7x_1 - 4x_2 + 12 \;=\; 0$ (from p.19)

$$\mathbf{x}^T \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x} - 2\mathbf{x}^T \begin{pmatrix} 7/2 \\ 2 \end{pmatrix} + 12 \;=\; 0$$

$A\mathbf{v} = \mathbf{b}$

can see it from the picture on p.19

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{v} \;=\; \begin{pmatrix} 7/2 \\ 2 \end{pmatrix} \quad \longrightarrow \quad \mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

put into the conic equation $\mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $\quad x_1 = 4\,, \quad x_2 = -1$

$$4^2 + 4 \cdot (-1) - 7 \cdot 4 - 4 \cdot (-1) + 12 \;=\; 16 - 4 - 28 + 4 + 12 \;=\; 0$$

center on coinc
$\rightsquigarrow$ conic degenerate

Example 7.6. Let's test this on a complicated empty conic

$$\big(\underbrace{x_1 + x_2 - 3}_{[1]}\big)^2 + \big(\underbrace{x_1 - 4}_{[2]}\big)^2 + 1 \;=\; 0 \tag{10}$$

$$2x_1^2 + 2x_1x_2 + x_2^2 - 14x_1 - 6x_2 + 26 \;=\; 0 \tag{11}$$

Now forget first equation; how do you see conic is empty?

$$\mathbf{x}^T \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \mathbf{x} - 2\mathbf{x}^T \underbrace{\begin{pmatrix} 7 \\ 3 \end{pmatrix}}_{\mathbf{b}} + 26 \;=\; 0$$

Let's first check $|A| \;=\; 2 \cdot 1 - 1 \cdot 1 = 1 > 0$ (ok)

Solve $\quad \underset{A}{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} \mathbf{v} \;=\; \underset{b}{\begin{pmatrix} 7 \\ 3 \end{pmatrix}} \quad \longrightarrow \quad \mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$\uparrow$ can see this also in (10) because the two lines given by $[1] = 0$ and $[2] = 0$ intersect at $(4, -1)$

Test l.h.s. of (11) for $\mathbf{x} = \mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$x_1 = 4,\; x_2 = -1$

$$\begin{array}{rcccccccl} t = 2\cdot 4^2 & +2\cdot 4\cdot(-1) & +(-1)^2 & -14\cdot 4 & -6\cdot(-1) & +26 & = & \overset{\text{clear from (10)}}{\downarrow} \\ 32 & -8 & +1 & -56 & +6 & +26 & = & 1 \end{array}$$

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad a = 2 > 0$$

$a \cdot t = 2 \cdot 1 > 0 \quad \rightsquigarrow \quad$ conic is empty $\qquad$ (End Example 7.6)

B To determine rotation, we need to find eigenvectors of A in (9) (rotation does not depend on $\mathbf{b}$, d)

1) If EV $\lambda_1 = \lambda_2$ (and conic is not degenerate) then conic is a circle, and rotation does not matter

$$\lambda_1 = \lambda_2 \, (=: \lambda) \quad \Longleftrightarrow \quad A = \lambda \cdot Id = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{scalar}$$

2) So assume $\lambda_1 \neq \lambda_2$

Let $\mathbf{v}_1$, $\mathbf{v}_2$ be $\underset{}{\overset{\|\mathbf{v}_i\| = 1}{\underline{\text{normalized}}}}$ eigenvectors to λ_1, λ_2.

$A\mathbf{v}_i = \lambda_i \mathbf{v}_i$. We know $\mathbf{v}_1^T \mathbf{v}_2 = 0$ $\quad$ $\|\mathbf{v}_i\| = 1$ determines $\mathbf{v}_i$ up to ± 1

as before left $R = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$.

We know R is orthogonal. Thus $|R| = 1$ or $|R| = -1$. If $|R| = -1$, replace $\mathbf{v}_1$ by $-\mathbf{v}_1$.

Thus assume $|R| = 1 \rightsquigarrow R = R_{\alpha}$ (rotation matrix).

We have

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$AR = R\Lambda$

$A = R\Lambda R^T \rightsquigarrow R$ is the applied rotation

compare to (5) on p.17 with $D = \Lambda$

$\underline{\alpha \text{ is the angle of rotation}}$
(up to multiples of $180°$)

Example 7.7. We consider the matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ of p.17.

$\det(A - \lambda Id) = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$

$\lambda_{1,2} = 3 \pm \sqrt{9-8} \qquad \lambda_1 = 4 \quad \lambda_2 = 2$

$\underline{\lambda_1 = 4} \qquad A_1 = A - \lambda_1 Id = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

$$A_1\mathbf{v}_1 = \mathbf{0} \quad \mathbf{v}_1 = \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} \qquad -v_{1,1} - v_{1,2} = 0 \longrightarrow$$

$$v_{1,2} = t,\ v_{1,1} = -t \quad \longrightarrow \mathbf{v} = t\begin{pmatrix} -1 \\ +1 \end{pmatrix}$$

Determine t using $\|\mathbf{v}_1\| = |t| \cdot \sqrt{2} = 1 \to t = \dfrac{1}{\sqrt{2}}$

$$\mathbf{v}_1 = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\underline{\lambda_2 = 2}$ $\quad A_2 = A - \lambda_2 Id = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$$A_2\mathbf{v}_2 = \mathbf{0} \quad \mathbf{v}_2 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \|\mathbf{v}_2\| = |t| \cdot \sqrt{2} = 1$$

$$\to \mathbf{v}_2 = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

choose first '+' in both $\pm$:

$$R = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ +1 & 1 \end{pmatrix} \qquad |R| = \frac{1}{2} \cdot (-2) = -1$$

change sign of $\mathbf{v}_1 \to \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \to$ thus now sign correct

$$R = R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \cos\alpha = \frac{1}{\sqrt{2}}$$

$$\sin\alpha = -\frac{1}{\sqrt{2}}$$

$\alpha = -45^\circ$

> Why did we get -45° and not 45°, as we did in example 7.2?
> Because we used different order of EV λ_i. With (6), we put in (8) $\lambda_1 = 2,\ \lambda_2 = 4$.
> If we use order $\lambda_1 = 2,\ \lambda_2 = 4$ of p.17 here, we have $R = \begin{pmatrix} \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
> $\rightsquigarrow \alpha = 45^\circ$, as in that example

determining <u>(symmetry) axes</u> (except for circle) and <u>asymptotes of hyperbola</u>

symmetry axes = $\mathbf{v}+$ multiples of eigenvectors of A (= cols of R) =
$= \mathbf{v} + t_1\mathbf{v}_1$, and $\mathbf{v} + t_2\mathbf{v}_2$ (for $t_{1,2} \in \mathbb{R}$ independent variables)

$$\text{asymptotes (of hyperbola)} = \mathbf{v} + t_\pm \cdot \underbrace{R \cdot \begin{pmatrix} \pm\sqrt{|\lambda_2|} \\ \sqrt{|\lambda_1|} \end{pmatrix}}_{\mathbf{v}_\pm}.$$

$(t_+,\ t_- \in \mathbb{R}$ independent, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_1\lambda_2 < 0)$.

Example

$x_1^2 + x_1x_2 - 7x_1 - 4x_2 + 12 \ = \ 0$ (from p.19)

$$\mathbf{x}^T \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x} - 2\mathbf{x}^T \begin{pmatrix} 7/2 \\ 2 \end{pmatrix} \dots$$

$$\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\lambda^2 - \lambda - 1/4 = 0 \qquad \lambda = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{4}} \ = \ \frac{1 \pm \sqrt{2}}{2}$$

In calculating asymptote vecs can use multiples, so write $\mathbf{w} \doteq \mathbf{u}$ for $\mathbf{w} = c \cdot \mathbf{u}$, $c \neq 0$.

$$\lambda_1 = \frac{1+\sqrt{2}}{2} \qquad A - \lambda_1 Id = \begin{pmatrix} \frac{1-\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1-\sqrt{2}}{2} \end{pmatrix} \rightsquigarrow \mathbf{v}_1 \doteq \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{2}}{2} \rightsquigarrow \mathbf{v}_2 \doteq \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \quad \text{(by } \perp\text{ity} \because \lambda_1 \neq \lambda_2 \text{ no circle!)}$$

$$\frac{1}{\sqrt{4-\sqrt{8}}} \begin{pmatrix} 1 & 1-\sqrt{2} \\ -1+\sqrt{2} & 1 \end{pmatrix} \ = \ R$$

$$\mathbf{v}_\pm \doteq \frac{1}{\sqrt{4-\sqrt{8}}} \begin{pmatrix} 1 & 1-\sqrt{2} \\ -1+\sqrt{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} \pm\sqrt{\frac{\sqrt{2}-1}{2}} \\ \sqrt{\frac{\sqrt{2}+1}{2}} \end{pmatrix} \doteq \begin{pmatrix} 1 & 1-\sqrt{2} \\ -1+\sqrt{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} \pm 1 \\ \sqrt{2}+1 \end{pmatrix}$$

$$= \begin{pmatrix} \pm 1 - 1 \\ \mp 1 \pm \sqrt{2} + \sqrt{2} + 1 \end{pmatrix} \ = \ \underset{-}{\begin{pmatrix} -2 \\ 2 \end{pmatrix}}, \underset{+}{\begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}} \doteq \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

giving $\begin{pmatrix} 4 \\ -1 \end{pmatrix} + \tilde{t}_- \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \tilde{t}_+ \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (\tilde{t}_\pm \doteq t_\pm),$

the lines given by the two factors $(x_1 + x_2 - 3)(x_1 - 4) \ = \ 0$

(the hyp. case degenerates into its asymptotes)

Degeneracy test for parabolic conics ($|A| = 0$)

We consider now (7)

$$(7) \qquad c_1x_1^2 + c_2x_2^2 + c_3x_1x_2 + c_4x_1 + c_5x_2 + c_6 \ = \ 0$$

and we assume that

$$A = \begin{pmatrix} c_1 & c_3/2 \\ c_3/2 & c_2 \end{pmatrix} \quad \text{has} \quad |A| = 0\,, \quad \text{but} \quad A \neq 0$$

This parabolic case determines either

$$\begin{array}{ll} \text{a parabola} & \text{(regular case)} \\ \left.\begin{array}{l} \text{2 parallel lines} \\ \text{1 line} \\ \varnothing \end{array}\right\} & \text{degenerate cases} \end{array}$$

$$|A| = c_1 c_2 - \frac{c_3^2}{4} = 0 \qquad \longrightarrow \quad c_1 c_2 \geqslant 0 \to c_1, c_2 \text{ have same sign (or one is 0)}$$

w.l.o.g. multiply (7) by -1 so that $c_1, c_2 \geqslant 0$.

$$\text{Let } \varepsilon = \begin{array}{l} \operatorname{sgn}(c_3) \text{ if } c_3 \neq 0 \\ \underline{\text{or 1 of } c_3 = 0} \end{array} \quad \Longrightarrow \quad c_3 = 2\varepsilon \cdot \sqrt{c_1 c_2}$$

Then $c_1 x_1^2 + c_2 x_2^2 + c_3 x_1 x_2 = \left(\sqrt{c_1}x_1 + \varepsilon\sqrt{c_2}x_2\right)^2$.

$$\text{Degeneracy test:} \quad c_4 \varepsilon \sqrt{c_2} = \sqrt{c_1} c_5 \ ? \qquad \Longleftrightarrow \qquad \begin{array}{l} c_4 x_1 + c_5 x_2 \\ \text{multiple of} \\ \sqrt{c_1}x_1 + \varepsilon\sqrt{c_2}x_2 \end{array}$$

If no $\to$ conic is parabola (regular)
if yes, let h be so that

$$h \cdot (\sqrt{c_1}x_1 + \varepsilon\sqrt{c_2}x_2) = c_4 x_1 + c_5 x_2 \quad \left(\begin{array}{c} h = \dfrac{c_4}{\sqrt{c_1}} \text{ or } h = \dfrac{c_5}{\varepsilon\sqrt{c_2}} \\ \text{whatever makes sense} \end{array} \right)$$

Test now the solvability (and number of solutions) of the equation

$$y^2 + hy + c_6 = 0$$

(Here $y = \sqrt{c_1}x_1 + \varepsilon\sqrt{c_2}x_2$.)

$$\begin{array}{lll} \dfrac{h^2}{4} > c_6 & \text{two solutions} & \longrightarrow \text{two parallel lines} \\ \dfrac{h^2}{4} = c_6 & \text{one solution} & \longrightarrow \text{one line} \\ \dfrac{h^2}{4} < c_6 & \text{no solution} & \longrightarrow \text{empty} \end{array}$$

Example 7.8. $-4x_1^2 - x_2^2 - 4x_1x_2 + 10x_1 + 5x_2 - 6 = 0$

$$A = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \qquad |A| = 0$$

here $c_1, c_2 < 0$ multiply by -1

$$4x_1^2 + x_2^2 + 4x_1x_2 - 10x_1 - 5x_2 + 6 = 0$$

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \qquad |A| = 0$$

$\varepsilon = \mathrm{sgn}(c_3) = 1$

$\sqrt{c_1} = 2,\ \sqrt{c_2} = 1 \quad \longrightarrow \quad 4x_1^2 + x_2^2 + 4x_1x_2 \ = \ (2x_1 + x_2)^2$

now apply degeneracy test: $\underset{c_4}{-10} \cdot \underset{\varepsilon}{1} \cdot \underset{\sqrt{c_2}}{1} \ = \ \underset{\sqrt{c_1}}{2} \cdot \underset{c_5}{(-5)}$?

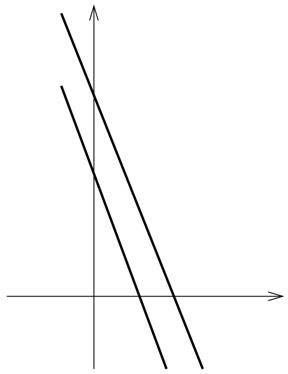
yes $\rightarrow$ $\underline{\overset{\text{conic}}{\text{degenerate}}}$

$$h \ = \ \frac{c_4}{\sqrt{c_1}} \ = \ -\frac{10}{2} \ = \ -5 \qquad (\text{ok since } \sqrt{c_1} \neq 0)$$

Thus $\quad 4x_1^2 + x_2^2 + 4x_1x_2 - 10x_1 - 5x_2 + 6$
$\quad = \ y^2 - 5y + 6 \quad (\text{with } y = 2x_1 + x_2)$

해
Solutions of $y^2 - 5y + 6 = 0$ are $y = 2$ and
$y = 3$.

(2 solutions because $6 < \left(\frac{5}{2}\right)^2$)
Thus we have <u>2 parallel lines</u> $\quad 2x_1 + x_2 = 2$ and
$2x_1 + x_2 = 3$



Example 7.9. $-4x_1^2 - x_2^2 - 4x_1x_2 + 10x_1 + 5x_2 - 6.25 \ = \ 0$

repeat calculation of previous example until

$$0 \ = \ y^2 - 5y + 6.25 \qquad (\text{with } y = 2x_1 + x_2)$$

$$6.25 = \left(\frac{5}{2}\right)^2 \longrightarrow \text{one solution } y = 2.5$$

conic $=$ <u>1 line</u> $\ 2x_1 + x_2 = 2.5$

Example 7.10. $-4x_1^2 - x_2^2 + 4x_1x_2 - 5x_1 - 6x_2 + 6$

$\rightarrow \quad 4x_1^2 + x_2^2 - 4x_1x_2 + 5x_1 + 6x_2 - 6$

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \qquad \varepsilon = \mathrm{sgn}(-2) = -1$$

$\sqrt{c_1} = 2 \quad \sqrt{c_2} = 1$

degeneracy test:

$$\underset{c_4}{5} \cdot \underset{\varepsilon}{(-1)} \cdot \underset{\sqrt{c_2}}{1} \ = \ \underset{c_5}{6} \cdot \underset{\sqrt{c_1}}{2} \quad \longrightarrow \quad \text{no} \quad \longrightarrow \quad \underline{\text{parabola}}$$

(regular)

Example 7.11. $4x_2^2 - 4x_2 + 2 = 0$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \qquad (c_1 = 0,\ c_2 \geqslant 0 \to \text{no need to change sign})$$

$$\operatorname{sgn}(0) = 0 \longrightarrow \varepsilon = 1$$

$$\sqrt{c_1} = 0 \quad \sqrt{c_2} = 2 \qquad\qquad c_4 = 0 \quad c_5 = -4$$

degeneracy test:

$$\underset{c_4}{0} \cdot \underset{\varepsilon}{1} \cdot \underset{\sqrt{c_2}}{2} = \underset{c_5}{-4} \cdot \underset{\sqrt{c_1}}{0} \longrightarrow \text{yes} \quad \text{degenerate}$$

$$h = \frac{c_4}{\sqrt{c_1}} = \frac{0}{0} ✘ \qquad h = \frac{c_5}{\varepsilon\sqrt{c_2}} = \frac{-4}{2} = -2$$

$$y^2 \underbrace{-\,2y}_{+hy} + \underbrace{2}_{c_6} \qquad \Big(y = \underbrace{\sqrt{c_1}}_{0}\, x_1 + \underbrace{\varepsilon}_{1}\ \underbrace{\sqrt{c_2}}_{2}\, x_2 = x_2 \Big)$$

$$c_6 > \frac{h^2}{4} \longrightarrow \text{no solution } y \to \underline{\text{conic} = \varnothing}$$

<u>determining vertex and (symmetry) axis of parabola</u>

$$\mathbf{x}^T R D R^T \mathbf{x} - 2\mathbf{x}^T \mathbf{b} + d = 0$$

$$A = RD\overset{\overset{\displaystyle R^T}{\shortparallel}}{R^{-1}}$$

$$RD = A\underset{\substack{\uparrow \\ \text{cols = eigenvecs of } A}}{R}$$

first col include Evec to $\lambda \neq 0$.

So $D = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$.

Let $\mathbf{x} = R\mathbf{y}$. Then

$$\underbrace{\mathbf{y}^T D \mathbf{y}}_{\lambda y_1^2} - 2\mathbf{y}^T R^T \mathbf{b} + d = 0$$

Solve for y_2 and find vertex $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and axis $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot t$.

(If you want vector pointing inside parabola, $\cdots + \begin{pmatrix} 0 \\ \operatorname{sgn}(z_2 \cdot \lambda) \end{pmatrix} \cdot t$ for $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = R^T \mathbf{b}$)

Then orig vertex is $\mathbf{x} = R \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and axis $R \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \overbrace{R \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{\text{Evec to } \lambda = 0} \cdot t$.

<u>Example</u>

$x_1^2 + 4x_1x_2 + 4x_2^2 + 5x_2 - 1 = 0.$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad \det A = 0.$$

$(\lambda - 1)(\lambda - 4) = 4 \Rightarrow \lambda = 0, 5$ EV.

$\lambda = 0 \quad \text{Evec} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot t$

$\lambda = 5 \quad \text{Evec} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot t$ (test: Evec $\perp$)

$$R = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \overset{\det = -1}{\rightsquigarrow} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = R_\theta = R$$

$$\theta = \arccos(\tfrac{1}{\sqrt{5}}) = \arctan(2)$$

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = R^T = R^{-1}.$$

rotate: $\mathbf{x} = R\mathbf{y}$

$$R^T\mathbf{b} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -5/2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -5 \\ -5/2 \end{pmatrix}$$

$$\mathbf{y}^T \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{y} - \frac{1}{\sqrt{5}} \mathbf{y}^T \begin{pmatrix} -10 \\ -5 \end{pmatrix} - 1 = 0.$$

$$5y_1^2 + \frac{10}{\sqrt{5}} y_1 + \frac{5}{\sqrt{5}} y_2 - 1 = 0.$$

$$y_2 = -\left(\sqrt{5} y_1^2 + 2y_1 - \frac{1}{\sqrt{5}}\right)$$ (can solve for $y_2 \rightsquigarrow$ is regular parabola)

$$-\left(\sqrt{5}\left(y_1 + \frac{1}{\sqrt{5}}\right)^2 - \frac{2}{\sqrt{5}}\right)$$

vrt/axis $\left(-\frac{1}{\sqrt{5}}, +\frac{2}{\sqrt{5}}\right) + (0,1) \cdot t$

rotate back:

$$R \cdot \left[\left(-\frac{1}{\sqrt{5}}, +\frac{2}{\sqrt{5}}\right) + (0,1) \cdot t\right] = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot \tilde{t} \qquad (\tilde{t} = t/\sqrt{5})$$

$$\frac{1}{5} \begin{pmatrix} -5 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot \tilde{t}$$

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot \tilde{t}$$