GS2004 **Linear Algebra** Fall 2019

Examples (3 problems)

Oct 07

1. Evaluating series

1.1. Partial fractions.

Problem 1. *(15 points)* Using the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,, \tag{$*$}$$

evaluate

$$\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^2}\,.$$

Solution. We start with the partial fraction decomposition

$$\frac{1}{(n^2-1)^2} = \frac{a}{n+1} + \frac{b}{(n+1)^2} + \frac{c}{n-1} + \frac{d}{(n-1)^2}\,. \tag{$\dagger$}$$

After multiplying by the denominators and comparing coefficients (in the degree-3 polynomial in n), we have the following equation system

$$\begin{array}{cccc|cc} a & b & c & d & & \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -2 & -1 & 2 & 0 & n \\ -1 & 1 & 1 & 1 & 0 & n^2 \\ 1 & 0 & 1 & 0 & 0 & n^3 \end{array}$$

Row-reduction (here I do from bottom to top) gives

$$\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 1 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array} \qquad \begin{array}{cccc|c} 0 & 0 & -4 & 0 & 1 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array} \qquad \begin{array}{c} c = -1/4 \\ d = 1/4 \\ b = 1/4 \\ a = 1/4 \end{array}\,.$$

Thus ($\dagger$) resolves to

$$\frac{1}{(n^2-1)^2} = \frac{1}{4}\left(\frac{1}{n+1} + \frac{1}{(n+1)^2} - \frac{1}{n-1} + \frac{1}{(n-1)^2}\right),$$

and we can evaluate, using a telescoping seres, and the series $(*)$,

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{1}{(n^2-1)^2} &= \frac{1}{4}\sum_{n=2}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)+\frac{1}{(n+1)^2}+\frac{1}{(n-1)^2}\\
&= \frac{1}{4}\left(\left(-1-\frac{1}{2}\right)+\frac{\pi^2}{6}+\left(\frac{\pi^2}{6}-1-\frac{1}{4}\right)\right)\\
&= \underline{\frac{\pi^2}{12}-\frac{11}{16}}\\
&\approx 0.134867\,.
\end{aligned}$$

□

1.2. **Fourier series.**

Problem 2. *(20 points)* Evaluate

$$\sum_{n=0}^{\infty}\frac{1}{(n^2+1)^2}\,.$$

Solution. We start with the identity

$$\frac{1}{a+in}-\frac{1}{-a+in} = \frac{-2a}{-n^2-a^2}\,.$$

Thus, setting $a=1$, we have

$$\frac{1}{n^2+1} = \frac{1}{2}\cdot\frac{1}{1+in}-\frac{1}{2}\cdot\frac{1}{-1+in}\,. \tag{‡}$$

Further recall that

$$\left\langle e^{at},e^{-int}\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{at}e^{int}\,dt = \frac{1}{2\pi}\cdot\frac{1}{a+in}\left(e^{2\pi a}-1\right)\,.$$

This suggests us to consider the Fourier series of the function

$$f(t) = \frac{\pi}{e^{2\pi}-1}e^{t}-\frac{\pi}{e^{-2\pi}-1}e^{-t}\,.$$

Then, using (‡), we have

$$\left\langle f,e^{int}\right\rangle = \frac{1}{(-n)^2+1} = \frac{1}{n^2+1}\,.$$

Parseval's identity gives on f

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+1)^2} &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\pi}{e^{2\pi}-1}e^{t}+\frac{\pi}{1-e^{-2\pi}}e^{-t}\right|^2\,dt \qquad \text{(now use } f(t)>0 \text{ on } [0,2\pi])\\
&= \frac{1}{2\pi}\left[\frac{\pi^2}{(e^{2\pi}-1)^2}\cdot\frac{1}{2}\cdot\left(e^{4\pi}-1\right)+\frac{\pi^2}{(1-e^{-2\pi})^2}\cdot\frac{1}{2}\cdot\left(1-e^{-4\pi}\right)+\frac{4\pi^3}{(e^{2\pi}-1)(1-e^{-2\pi})}\right]\\
&= \frac{\pi}{2}\cdot\frac{e^{2\pi}+1}{e^{2\pi}-1}+\frac{2\pi^2}{e^{2\pi}-2+e^{-2\pi}} \qquad \text{(the first two summands above are equal)}\\
&= \frac{\pi}{2}\cdot\frac{e^{2\pi}+1}{e^{2\pi}-1}+\frac{2\pi^2e^{2\pi}}{(e^{2\pi}-1)^2}\,.
\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(n^2+1)^2} &= \frac{1}{2}\left(1 + \sum_{n=-\infty}^{\infty} \frac{1}{(n^2+1)^2}\right) \\ &= \underline{\frac{1}{2} + \frac{\pi}{4}\cdot\frac{e^{2\pi}+1}{e^{2\pi}-1} + \frac{\pi^2 e^{2\pi}}{(e^{2\pi}-1)^2}} \\ &\approx 1.306837\,.\end{aligned}$$

□

Problem 3. *(20 points)* Evaluate

$$\sum_{n=0}^{\infty} \frac{1}{n^4+n^2+1}\,.$$

Solution. First,

$$\frac{1}{n^4+n^2+1} = \frac{1}{\left(n^2+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,. \tag{§}$$

We try the ansatz, for constants $a, b, A, B \in \mathbb{C}$ to find, and all $n \in \mathbb{Z}$,

$$\frac{A}{a+in} + \frac{B}{b+in} = \frac{A(b+in)+B(a+in)}{(a+in)(b+in)} = \frac{1}{-\left(n^2+\frac{1}{2}\right) - \frac{\sqrt{3}}{2}i}\,. \tag{¶}$$

Comparing denominators gives the two equations

$$\begin{aligned} ab - n^2 &= -n^2 - \frac{1}{2} - \frac{\sqrt{3}}{2}i \\ a+b &= 0 \end{aligned}$$

Thus,

$$ab = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \quad \text{and} \quad a = -b\,,$$

and

$$a^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad a = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

is *one* solution (this is enough). Then, comparing numerators in (¶) (with $b = -a$) gives

$$A\cdot(-a+in) + B(a+in) = 1\,,$$

thus

$$\begin{aligned} -A+B &= \frac{1}{a} \\ A+B &= 0 \end{aligned}$$

solving to

$$A = -\frac{1}{2a} = -\frac{\sqrt{3}}{4} + \frac{i}{4} \quad \text{and} \quad B = \frac{1}{2a} = \frac{\sqrt{3}}{4} - \frac{i}{4}\,.$$

Thus the ansatz (¶) resolves to

$$\frac{-\frac{\sqrt{3}}{4}+\frac{i}{4}}{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)+in}+\frac{\frac{\sqrt{3}}{4}-\frac{i}{4}}{\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)+in}=\frac{1}{-\left(n^2+\frac{1}{2}\right)-\frac{\sqrt{3}}{2}i}. \qquad (\|)$$

Further recall that

$$\left\langle e^{at},e^{-int}\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{at}e^{int}\,dt = \frac{1}{2\pi}\cdot\frac{1}{a+in}\left(e^{2\pi a}-1\right).$$

This suggests us to consider the Fourier series of the function

$$f(t) = 2\pi\left[\frac{-\frac{\sqrt{3}}{4}+\frac{i}{4}}{e^{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\cdot 2\pi}-1}e^{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\cdot t}+\frac{\frac{\sqrt{3}}{4}-\frac{i}{4}}{e^{\left(\frac{-\sqrt{3}}{2}-\frac{i}{2}\right)\cdot 2\pi}-1}e^{\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\cdot t}\right],$$

which can be simplified a bit:

$$f(t) = -\frac{\pi}{2}\left[\frac{-\sqrt{3}+i}{e^{\sqrt{3}\pi}+1}e^{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\cdot t}+\frac{\sqrt{3}-i}{e^{-\sqrt{3}\pi}+1}e^{\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\cdot t}\right].$$

Using (‖) and (§), we have

$$\left|\left\langle f,e^{int}\right\rangle\right|^2 = \frac{1}{(-n)^4+(-n)^2+1} = \frac{1}{n^4+n^2+1}.$$

Parseval's identity gives on f

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{1}{n^4+n^2+1} &= ||f||^2\\
&= \frac{\pi}{8}\int_0^{2\pi}\left|\frac{-\sqrt{3}+i}{e^{\sqrt{3}\pi}+1}e^{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\cdot t}+\frac{\sqrt{3}-i}{e^{-\sqrt{3}\pi}+1}e^{\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\cdot t}\right|^2 dt\\
&= \frac{\pi}{8}\left[\frac{4}{\left(e^{\sqrt{3}\pi}+1\right)^2}\,\frac{e^{2\sqrt{3}\pi}-1}{\sqrt{3}}+\frac{4}{\left(e^{-\sqrt{3}\pi}+1\right)^2}\,\frac{e^{-2\sqrt{3}\pi}-1}{-\sqrt{3}}+\right.\\
&\qquad\left. 2\int_0^{2\pi}\Re e\left(\overbrace{\frac{(-\sqrt{3}+i)(\sqrt{3}+i)}{(e^{\sqrt{3}\pi}+1)(e^{-\sqrt{3}\pi}+1)}}^{=:\,\alpha}\;\overbrace{e^{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\cdot t+\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\cdot t}}^{e^{it}}\right)dt\right].
\end{aligned}$$

Now, the first two summands above are equal. Moreover, the integrand in the third summand is of the form $\Re e\alpha\cos t-\Im m\alpha\sin t$, and thus the integral is 0. Then,

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^4+n^2+1} = \frac{\pi}{\sqrt{3}}\cdot\frac{e^{\sqrt{3}\pi}-1}{e^{\sqrt{3}\pi}+1}.$$

Thus

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n^4+n^2+1} &= \frac{1}{2}\left(1+\sum_{n=-\infty}^{\infty}\frac{1}{n^4+n^2+1}\right)\\ &= \underline{\frac{1}{2}+\frac{\pi}{2\sqrt{3}}\cdot\frac{e^{\sqrt{3}\pi}-1}{e^{\sqrt{3}\pi}+1}}\\ &\approx 1.39907\,.\end{aligned}$$

□