Exercises

1. Prove the following "laws of algebra" for $\mathbb{R}$, using only axioms (1)–(5):
   (a) If $x + y = x$, then $y = 0$.
   (b) $0 \cdot x = 0$. [*Hint:* Compute $(x + 0) \cdot x$.]
   (c) $-0 = 0$.
   (d) $-(-x) = x$.
   (e) $x(-y) = -(xy) = (-x)y$.
   (f) $(-1)x = -x$.
   (g) $x(y - z) = xy - xz$.
   (h) $-(x + y) = -x - y$; $-(x - y) = -x + y$.
   (i) If $x \neq 0$ and $x \cdot y = x$, then $y = 1$.
   (j) $x/x = 1$ if $x \neq 0$.
   (k) $x/1 = x$.
   (l) $x \neq 0$ and $y \neq 0 \Rightarrow xy \neq 0$.
   (m) $(1/y)(1/z) = 1/(yz)$ if $y, z \neq 0$.
   (n) $(x/y)(w/z) = (xw)/(yz)$ if $y, z \neq 0$.
   (o) $(x/y) + (w/z) = (xz + wy)/(yz)$ if $y, z \neq 0$.
   (p) $x \neq 0 \Rightarrow 1/x \neq 0$.
   (q) $1/(w/z) = z/w$ if $w, z \neq 0$.
   (r) $(x/y)/(w/z) = (xz)/(yw)$ if $y, w, z \neq 0$.
   (s) $(ax)/y = a(x/y)$ if $y \neq 0$.
   (t) $(-x)/y = x/(-y) = -(x/y)$ if $y \neq 0$.
2. Prove the following "laws of inequalities" for $\mathbb{R}$, using axioms (1)–(6) along with the results of Exercise 1:
   (a) $x > y$ and $w > z \Rightarrow x + w > y + z$.
   (b) $x > 0$ and $y > 0 \Rightarrow x + y > 0$ and $x \cdot y > 0$.
   (c) $x > 0 \Leftrightarrow -x < 0$.
   (d) $x > y \Leftrightarrow -x < -y$.
   (e) $x > y$ and $z < 0 \Rightarrow xz < yz$.
   (f) $x \neq 0 \Rightarrow x^2 > 0$, where $x^2 = x \cdot x$.
   (g) $-1 < 0 < 1$.
   (h) $xy > 0 \Leftrightarrow x$ and y are both positive or both negative.
   (i) $x > 0 \Rightarrow 1/x > 0$.
   (j) $x > y > 0 \Rightarrow 1/x < 1/y$.
   (k) $x < y \Rightarrow x < (x + y)/2 < y$.
3. (a) Show that if $\mathcal{A}$ is a collection of inductive sets, then the intersection of the elements of $\mathcal{A}$ is an inductive set.
   (b) Prove the basic properties (1) and (2) of $\mathbb{Z}_+$.
4. (a) Prove by induction that given $n \in \mathbb{Z}_+$, every nonempty subset of $\{1, \ldots, n\}$ has a largest element.
   (b) Explain why you cannot conclude from (a) that every nonempty subset of $\mathbb{Z}_+$ has a largest element.

is also a finite intersection of elements of $\mathcal{S}$, so it belongs to $\mathcal{B}$.

Exercises

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.
2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.
3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

   a topology on X?
4. (a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
   (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
   (c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.
5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.
6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.
7. Consider the following topologies on $\mathbb{R}$:

$$\begin{aligned}
\mathcal{T}_1 &= \text{the standard topology,}\\
\mathcal{T}_2 &= \text{the topology of } \mathbb{R}_K,\\
\mathcal{T}_3 &= \text{the finite complement topology,}\\
\mathcal{T}_4 &= \text{the upper limit topology, having all sets } (a, b] \text{ as basis,}\\
\mathcal{T}_5 &= \text{the topology having all sets } (-\infty, a) = \{x \mid x < a\} \text{ as basis.}
\end{aligned}$$

   Determine, for each of these topologies, which of the others it contains.
8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$